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Fukaya category for Landau-Ginzburg orbifolds and Berglund-Hübsch conjecture for invertible curve singularities

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Abstract

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From a fixed cohomology class $\Gamma \in SH^\bullet(M)$ of a Liouville manifold M, we construct a new A_∞ category denoted by $\mathscr{C}_\Gamma$ on which the quantum cap action of $\Gamma : CW^\bullet(L,L) \to CW^\bullet(L,L)$ vanishes homotopically.

With this construction on one hand, we consider a symplectic Landau - Ginzburg model (W,G) defined by a weighted homogeneous polynomial W and its symmetry group G. From wrapped Fukaya category and a monodromy information of the Milnor fiber, we construct a new Fukaya category $\mathscr{F}(W,G)$ for each pair (W,G) on which the monodromy action vanishes. It is a symplectic analogue of the variation operator in singularity theory.

We also show that the mirror of the monodromy action is a restriction of a mirror Landau-Ginzburg model to a certain hypersurface. As an application, we prove Berglund-Hübsch homological mirror symmetry for all invertible curve singularities.

Key words: Lagrangian Floer theory, Mirror symmetry, Orbifold, Invertible polynomials, Matrix factorization
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Chapter 1

Introduction

Given a singularity W which is a polynomial in $\mathbb{C}^n$, Fukaya-Seidel category is defined by perturbing W into a Morse function W_ϵ and considering the collection of vanishing cycles of W_ϵ and their directed Fukaya A_∞-category. This has been one of the central topic in symplectic geometry and mirror symmetry. In particular, Fukaya-Seidel category defines a symplectic category for a Landau-Ginzburg model W in the setting of homological mirror symmetry conjecture. Namely, if W has a mirror complex manifold M, then Fukaya-Seidel category should be derived equivalent to the derived category of coherent sheaves on M. If W has a mirror Landau-Ginzburg orbifold $(\widehat{W}, H)$ in the sense that $\widehat{W}$ is H-invariant for a finite group H, then FS category should be derived equivalent to maximally graded category of matrix factorization of $\widehat{W}$. Many instances of such homological mirror symmetry has been proved.

In spite of its importance, Fukaya-Seidel category for Landau-Ginzburg orbifold has not been known. The main difficulty is that when a finite group G acts on $\mathbb{C}^n$ and W is G-invariant, its perturbation to a Morse function W_ϵ destroys the original symmetry G. Namely, W_ϵ is not G-invariant in general, and it has not been known how to overcome this difficulty.

In this paper, we introduce a different approach to define a Fukaya category of the singularity W when W is a weighted homogeneous polynomial. In this approach, we will not perturb W and hence we can define an equivariant version of it as well. Namely, for a diagonal symmetry group G of W, we are able to define a new Fukaya category for a Landau-Ginzburg orbifold (W, G).

Instead of using vanishing cycles(which needs perturbation), we will use wrapped Fukaya category of Milnor fiber and monodromy map. Let us recall standard classical singularity theory for analogy. Given an isolated singularity $W : \mathbb{C}^n \to \mathbb{C}$ at the origin, a monodromy map in classical singularity theory is defined by considering the parallel transports along a circle centered at the origin in $\mathbb{C}$. In particular, for the Milnor fiber $X = W^{-1}(1)$, there is a variation map $\mathrm{var} : H_{n-1}(X, \partial X) \to H_{n-1}(X)$ given by the difference of the cycle and its image under monodromy. The image of variation map for Morse singularity are vanishing cycles. Our approach is to use the Milnor fiber and monodromy information to define a Fukaya category, instead of vanishing cycles.

Symplectic cohomology is a version of Hamiltonian Floer cohomology for Liouville domains introduced by Cieliebak, Floer and Hofer [CFH95] and Viterbo [Vit99]

With this new definition of Fukaya category for a Landau-Ginzburg orbifold (W,G), we formulate and prove homological mirror symmetry between invertible curve singularities, called Berglund-Hübsch HMS conjecture.

Berglund-Hübsch introduced mirror pairs for invertible singularities. $W : \mathbb{C}^n \to \mathbb{C}$ is called invertible singularity if W has n-terms and its $n \times n$ exponent matrix E is non-degenerate. Let G be a diagonal symmetry group of W, and let G_W be the maximal diagonal symmetry group of W, which is a finite abelian group. Then, Berglund-Hübsch dual of (W,G) is given by (W^T, G^T) where W^T is another invertible singularity with exponent matrix E^T and $G^T = \mathrm{Hom}(G_W/G, U(1))$. If G is trivial, G^T becomes G_{W^T}, the maximal diagonal symmetry group for the mirror singularity.

The following version of Berglund-Hübsch HMS conjecture has been proved.

$$\text{Fukaya Seidel category of } W \;\longleftrightarrow\; MF^{\text{max. gr}}(W^T)$$

In this paper, we prove the following complete form of Berglund-Hübsch HMS conjecture

$$\text{Fukaya category of } (W,G) \;\longleftrightarrow\; MF(W^T, G^T)$$

Our proof is constructive and geometric in the sense that we start with (W,G) and we obtain the mirror pair (W^t, G^t) via Floer theory of (W,G), and homological mirror symmetry A_∞-functor is also constructed geometrically.

When $G = G_W$ is maximal, G^T is trivial group and hence right hand side of the above Berglund-Hübsch HMS is a $\mathbb{Z}/2$-graded matrix factorization category of W^T.

The ring $S := \mathbb{C}[x_1, \cdots, x_n]/(W^T)$ is a Cohen-Macaulay ring, and maximal Cohen-Macaulay modules of S has been studied intensively in the 80's. In particular, Eisenbud showed that maximal Cohen-Macaulay modules are equivalent to $\mathbb{Z}/2$-graded matrix factorizations [Eis80].

When W^T is ADE singularity, it is known that there are only finitely many indecomposable objects, and irreducible morphisms between them, and such an information is recorded in the Auslander-Reiten quiver.

Our work provides a geometric interpretation of such Auslander-Reiten quiver. Namely, in the above BH HMS correspondence, we specify non-compact Lagrangians that are mapped to each indecomposable object. Moreover, exact sequences in the Auslander-Reiten quiver can be interpreted as a surgery exact sequence between Lagrangian submanifolds.

Kreuzer-Sharke [KS92] classified invertible singularities in n-variables and have shown that they are given by Thom-Sebastiani sums of Fermat, Chain and Loop type singularities. Thus we may consider the following three families.

- (Fermat type) $F_{p,q} = x^p + y^q$
- (Chain type) $C_{p,q} = x^p + xy^q$
- (Loop type) $L_{p,q} = x^p y + xy^q$

The way that mirror polynomial W^T arise from Floer theory of (W, G_W) is quite interesting. One may consider the mirror of the Milnor fiber M_W of W, its maximal symmetry group G_W, and Fukaya category of an orbifold $[M_W/G_W]$ without considering the monodromy map. In this case, we find that we have $\widetilde{W} : \mathbb{C}^3 \to \mathbb{C}$, which are of the form

$$x^p + y^q + xyz, y^q + xyz, xyz$$

for Fermat, Chain, Loop cases, which is not the transpose mirror polynomial W^T.

But if we restrict $\widetilde{W}$ to a certain graph hypersurface $g(x,y,z)=0$ then we obtain the transpose polynomial W^T. Namely, if we set

$$z=0, z-x^{p-1}=0, z-x^{p-1}-y^{q-1}=0.$$

we obtain the mirror polynomials as expected for every cases:

$$x^p+y^q, x^p y+y^q, x^p y+xy^q.$$

We find that this restriction $g(x,y,z)=0$ comes from the monodromy information of the singularity. Namely, there is a distinguished degree zero symplectic cohomology class Γ_W, which come from family of Reeb chords on a quotient orbifold of a Milnor fiber $[\partial M_W/G_W]$. These Reeb chords are exactly the monodromy map around the origin for weighted homogeneous polynomials. We prove that closed-open string map from symplectic cohomology of $[M_W/G_W]$ to Jacobian ring of $\widetilde{W}$ exists and it maps Γ to $g(x,y,z)$.

We show that the natural functor from $\mathscr{W}([M_W/G_W])$ to the new A_∞-category $\mathscr{C}_\Gamma$ is mirror to the natural restriction map from $MF(\widetilde{W})$ to $MF(W^T)$. Furthermore, we construct an A_∞-functor from $\mathscr{C}$ to $MF(W^T)$ and show that this is an A_∞-quasi-isomorphism. This proves the Berglund-Hübsch HMS conjecture for the case of maximal diagonal symmetry group (W,G_W) for the A-side. For a subgroup $G\subset G_W$, such an A_∞-functor can be lifted to an equivariant version, proving the rest of the Berglund-Hübsch HMS conjecture.

Chapter 2

Basic Floer theory

In this section, we describe a general Floer theory.

2.1 Liouville manifold with cylindrical end

Our basic object of study will be a Liouville manifold.

Definition 2.1.1. *A Liouville manifold is a symplectic manifold (M,ω) with a one form λ called Liouville form, such that*

$$d\lambda = \omega.$$

The Liouville vector field Z is a symplectic dual of λ.

$$i_Z\omega = \lambda$$

A Liouville manifold M is to have a cylindrical end if there is a compact submanifold $M_{cpt} \subset M$ such that M is the symplectization of M_{cpt}

$$M = M_{cpt} \bigcup_{\partial M_{cpt}} \partial M_{cpt} \times [1,\infty).$$

A subset $\partial M_{cpt} \times [1,\infty]$ is called a (cylindrical) end of M. The flow Z should be *transverse to* ∂M_{cpt} *and it is of the form*

$$Z = r\frac{\partial}{\partial r}.$$

at the cylindrical end.

To simplify the rest of the discussion, we will assume that our Liouville manifold with cylindrical end M have real dimension $2n$ and satisfies

$$c_1(TM) = 0.$$

We will denote

$$\psi^t$$

as a Liouville flow of time $\log t$.

It is easy to check that the restriction $\Lambda = \lambda|_{M_{cpt}}$ is a contact form and the Liouville one form at the end is its rescaling

$$\lambda = r\Lambda$$

Then an automorphism $\phi : M \to M$ of such manifold is given by a automorphism of the contact manifold times identity at the end. A Reeb vector field R is defined by

$$R \in ker(d\Lambda), \quad \Lambda(R) = 1.$$

The following restrictions on Hamiltonians and almost complex structures are standard.

- We will work with a function $H \in C^\infty(M,\mathbb{R})$ such that $H > 0$, C^2-small on M_{cpt} and *quadratic at infinity*;

  $$H(x,r) = ar^2 + b \ \ (a > 0), \quad r \text{ is a coordinate of } [1,\infty)$$

  We denote the class of such function by $\mathscr{H}(M)$

- Whenever we consider a time dependent perturbation $H_{S^1} = H + F : S^1 \times M \to \mathbb{R}$, we assume $H_{S^1} > 0$, C^2-small on M_{cpt} so that the time-1 periodic orbit of H_{S^1} are non-degenerate. This is true for a generic perturbation.

- an almost complex structure J is called *c-rescaled contact type* if

$$-\frac{c}{r}\lambda \circ J = dr$$

at the end. We denote a class of such almost complex structure by $\mathcal{J}_c(M)$.

Next, we describe a Lagrangian submanifold we want to use. It is a collection $\mathcal{W}$ of exact properly embedded Lagrangian submanifolds, which may not be compact, but satisfies;

$$\text{Liouville one form } \lambda \text{ vanishes on } L \cap \partial M_{cpt} \times [1, \infty).$$

It means that the intersection $L \cap \partial M_{cpt}$ is a Legenderian submanifold, and L is **conical at the end**, i.e

$$L = \left(L \cap M_{cpt}\right) \bigcup \partial \left(L \cap M_{cpt}\right) \times [1, \infty).$$

Furthermore, all such L is required to have vanishing relative first Chern class $2c_1(M, L)$. We attach a spin structure and a grading function on each L. All Lagrangian submanifold we consider will implicitly carry these extra data.

2.2 Degree and index of Hamiltonian orbits and chords

Fix a small, time dependent perturbation $H_{S^1} : S^1 \times M \to \mathbb{R}$ of H. Let

$$\mathcal{O} := \mathcal{O}(M, H_{S^1})$$

be a set of time-1 orbits of S^1-dependent hamiltonian function H_{S^1}. We may assume all orbits are nondegenerate, which is true for a generic choice of F. For each $\gamma \in \mathcal{O}$, we trivialize $\gamma^* TM$ so that it induces the same homotopy class of existing trivial bundle $\gamma^* K_M$. Then, the derivative of a hamiltonian flow $d\phi_{H_{S^1}}$ restricted on γ induces a path of symplectic matrix Φ_t. Let B_t be a path of symmetric matrix which satisfies a differential equation

$$\frac{d}{dt}\Phi_t = J \cdot B_t \cdot \Phi_t.$$

Here, J is a standard complex structure of $\mathbb{C}^n$. Equip $\mathbb{C}$ a negative cylindrical coordinates

$$\mathbb{R}\times S^1 \to \mathbb{C} \tag{2.2.1}$$
$$(s,t) \mapsto e^{-s-2\pi i t}. \tag{2.2.2}$$

Fix any map $B \in C^\infty(\mathbb{C}, Mat_{n\times n}(\mathbb{C}))$ such that

$$B(s,t) = J\cdot B_t$$

for $s << 0$. Now define an operator

$$D_\Phi : W^{1,p}\left(\mathbb{C},\mathbb{C}^n\right) \to L^p\left(\mathbb{C},\mathbb{C}^n\right) \tag{2.2.3}$$
$$D_\Phi(X) = \partial_s X + J\cdot\partial_t X + B_t X \tag{2.2.4}$$

This is Fredholm because we have assumed that γ is nondegenerate.

Definition 2.2.1. *An orientation line o_γ associated to a hamiltonian orbits γ is defined as a determinant line of a Fredholm operator;*

$$\mathrm{Det}D_\Phi = \mathrm{Det}(\mathrm{KerD}_\Phi)\otimes\mathrm{Det}(\mathrm{CokerD}_\Phi)^\vee.$$

A degree of o_γ is defined as an index

$$\deg o_\gamma := \mathrm{indD}_\Phi = \dim_{\mathbb{R}}\mathrm{KerD}_\Phi - \dim_{\mathbb{R}}\mathrm{CokerD}_\Phi$$

This integer is also called cohomological Conley-Zehnder index in [A^+ 12].

An index can be computed topologically in the following way. A *crossing* $t\in(0,1)$ is a time when $\mathrm{Det}(\Phi_t - \mathrm{Id}) = 0$. A *Robbin-Salamon index* of γ is defined as

$$\mu_{RS}(\gamma) = \frac{1}{2}Sgn(B_0) + \sum_{t\in\text{crossing}} Sgn(B_t) + \frac{1}{2}Sgn(B_1).$$

It is known that

$$\deg o_\gamma = n - \mu_{RS}(\gamma).$$

We move on to a hamiltonian chords. For $L_0, L_1 \in \mathscr{W}$, we define

$$\chi(L_0, L_1; H)$$

to be the set of time-1 hamiltonian chords of H from L_0 to L_1. Let's trivialize $a^* TM$ so that it induces a same relative homotopy class of an existing trivial bundle $s^* K_M$. Then, a chord $x \in \chi(L_0, L_1)$ can be thought as a path of symplectic matrix Ψ_t with Lagrangian boundary conditions TL_i respectively. We obtain a path of symplectic matrix, still denoted by B_t, in a similar fashion. Now, equip $\mathbb{H}$ the following parametrization

$$(-\infty, 0] \times [0,1] \to \mathbb{H} \tag{2.2.5}$$
$$(s,t) \mapsto e^{-\pi s - 2\pi i t + \pi i}. \tag{2.2.6}$$

Also, choose a family of Lagrangian subspaces F_t such that $F_{s\times\{0\}} = TL_0$ and $F_{s\times\{1\}} = TL_1$. It is uniquely defined (up to homotopy) since our Lagrangian submanifold has gradings. Fix any map $B \in C^\infty(\mathbb{H}, Mat_{n\times n}(\mathbb{C}))$ such that

$$B(s,t) = J \cdot B_t$$

for $s << 0$. Now define an operator

$$D_\Psi : W^{1,p}\left(\mathbb{H}, \mathbb{C}^n, F_t\right) \to L^p\left(\mathbb{H}, \mathbb{C}^n\right) \tag{2.2.7}$$
$$D_\Psi(X) = \partial_s X + J \cdot \partial_t X + B_t X \tag{2.2.8}$$

This is Fredholm because we have assumed that a is nondegenerate.

Definition 2.2.2. *An orientation line o_x associated to a hamiltonian chords x is defined as a determinant line of a Fredholm operator;*

$$\mathrm{Det} D_\Psi = \mathrm{Det}(\mathrm{Ker D_\Psi}) \otimes \mathrm{Det}(\mathrm{Coker D_\Psi})^\vee.$$

A degree of o_x is defined as an index

$$\deg o_x := \mathrm{ind D_\Psi} = \dim_\mathbb{R} \mathrm{Ker D_\Psi} - \dim_\mathbb{R} \mathrm{Coker D_\Psi}$$

This integer is called Maslov index of x, and it coincides with the Maslov index

$\mu_M(x)$ of Lagrangian path x.

2.3 Moduli space of pseudo-holomorphic curves

We briefly describe a perturbation scheme for general moduli spaces of pseudo-holomorphic curves we use. We refer [Sei08], [AS10], [A$^+$12] and [Gan13] from which most of the material has been borrowed.

Let

$$S_{m_1,m_2;n_1,n_2}$$

be a moduli space of holomorphic discs D with m_1 positive interior markings, m_2 negative interior markings, n_1 positive boundary markings, and n_2 negative boundary marking. In this paper, we will only consider when there is only one positive markings.

$$S_{m;n,1} := S_{m,0;n,1}, \text{ or } S_{m,1;n} := S_{m,1;n,0}$$

The Deligne-Mumford compactification of this moduli space is denoted by $\overline{S}_{m_1,m_2;n_1,n_2}$ Let us denote

- $Z_+ = [0,\infty) \times [0,1]$ with coordinate (s,t)
- $Z_- = (-\infty,0] \times [0,1]$ with coordinate (s,t)
- $C_+ = [0,\infty) \times S^1$ with coordinate (s,t)
- $C_- = (-\infty,0] \times S^1$ with coordinate (s,t).

Definition 2.3.1. *A collection of strip and cylinder data for $S \in S_{m;n,1}$ or $S \in S_{m,1;n}$ is a choice of*

- *Strip-like ends $\epsilon^k_\pm : Z_\pm \to S$ which models a boundary marking x_k*
- *cylindrical ends $\delta^l_\pm : C_\pm \to S$ which models an interior marking y_k*

Such collection is said to be weighted if each strip and cylinder is endowed with a positive real number

- *$w^\pm_{S,k}$ for each strip-like end $\epsilon^k_\pm$*

- $v^{\pm}_{S,l}$ *for each strip-like end* $\delta^l_{\pm}$

Definition 2.3.2. *Let* $(S,\mathfrak{S})$ *denote a holomorphic disc* S *with a collection of weighted strip and cylinder data* $\{\kappa\}$ *with weight* $\{\nu_\kappa\}$.

1. *A one-form* α_S *is said to be compatible to* $\mathfrak{S}$*,*

$$\kappa^*\alpha_S = \nu_\kappa dt, \quad \forall\kappa$$

2. *An* $\mathfrak{S}$*-adapted rescaling function is a function* $a_S : S \to [1,\infty)$ *such that*

$$\kappa^* a_S = \nu_\kappa, \quad \forall\kappa$$

3. *For a fixed hamiltonian* $H \in \mathscr{H}(M)$*, an* $S-$*dependent Hamiltonian* H_S *is said to be compatible with* $(S,\mathfrak{S},H)$ *if*

$$\kappa^* H_S = \frac{H\circ\psi^{\nu_\kappa}}{\nu_\kappa^2}, \quad \forall\kappa$$

4. *For a fixed* S^1*-dependent almost complex structure* J_t*, an* $S-$*dependent almost complex structure* J_S *is called* $(S,\mathfrak{S},a_S,J_t)$ *- adpted if the following two conditions are satisfied*

$$J_p \in \mathscr{J}_{a_S(p)}, \quad \forall p\in S$$

$$\kappa^* J_S = \left(\phi^{\nu_\kappa}\right)^* J_t, \quad \forall\kappa$$

Finally, we define

Definition 2.3.3. *For a fixed disc S, a Floer data* F_S *consists of*

1. *A collection of weighted strip and cylinder data* $\mathfrak{S}$*;*
2. *a one form* α_S *compatible with* $(S,\mathfrak{S})$ *which is sub-closed, i.e,*

$$d\alpha_S \le 0$$

3. *An* $(S,\mathfrak{S})-$*adapted rescaling function* a_S*;*

4. *An S-dpendent, $(S, \mathfrak{S}, H)$-compatible hamiltonian H_S;*

5. *An S-dependent, $(S, \mathfrak{S}, a_S, J_t)$-adapted almost complex structure J_S.*

Also, we say F_S^1 and F_S^2 are conformally equivalent if F_S^2 is a rescaling by Liouville flow of F_S^1, up to constant ambiguity in the Hamiltonian terms.

A universal and consistent choice of Floer data is a choice of Floer data F_S for all $S \in S_{m;n,1}$ or $S \in S_{m,1;n}$ which varies smoothly over the moduli space. Since the space of Floer data is contractible, we can extend it to $\overline{S}_{m;n,1}$ or $\overline{S}_{m,1;n}$.

Example 2.3.4. *In the simplest case of a strip $S \in S_{0,1,1}$ or a cylinder $S \in S_{1,1;0}$, we choose a canonical strip-like/cylindrical end with weights* 1 *for all ends. A form* dt *is a compatible sub-closed one form.*

Definition 2.3.5. *Let $\gamma_i \in \mathcal{O}$ be an time-1 Hamiltonian orbits and $a_j \in \chi(L_{j-1}, L_j)$, $j = 1, \dots, n$ and $a_0 \in \chi(L_n, L_0)$ be Hamiltonian chords. Define*

$$\overline{\mathcal{M}}_{m;n,1}(\gamma_1, \dots, \gamma_m; a_1, \dots, a_n, a_0) \tag{2.3.1}$$

a space of maps

$$\left\{u : S \to M : S \in \overline{S}_{m;n,1}\right\} \tag{2.3.2}$$

satisfying the inhomogeneous Cauchy-Riemann equation with respect to J_S

$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{2.3.3}$$

and the following asymptotic/boundary conditions;

$$\lim_{s\to\infty} u \circ \epsilon_-^k(s, \cdot) = a_k \tag{2.3.4}$$

$$\lim_{s\to-\infty} u \circ \epsilon_+^0(s, \cdot) = a_0 \tag{2.3.5}$$

$$\lim_{s\to\infty} u \circ \delta_+^l(s, \cdot) = \gamma_l \tag{2.3.6}$$

$$u(z) \in \psi^{a_S(z)} L_i, \quad z \in \partial_i S, \text{ an } i\text{-th boundary component of } S. \tag{2.3.7}$$

We define $\overline{\mathcal{M}}_{m,1;n}(\gamma_1, \dots, \gamma_m; \gamma_0; a_1, \dots, a_n)$ in a similar fashion.

Here, we implicitly use that Liouville flow induces a $1-1$ correspondence between hamiltonian chords

$$\chi(L_0, L_1; H) \simeq \chi\left(\phi^t L_0, \phi^t L_1; (\phi^t)^* \left(\frac{H}{t}\right)\right).$$

The following compactness and transversality result is standard.

Lemma 2.3.6. *For a generic choice of universal and consistent Floer data,*

1. *The moduli spaces* $\overline{\mathcal{M}}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ *are compact.*

2. *For a given input* $\gamma_i, \quad i = 1, \ldots, m$ *and* $a_j, \quad j = 1, \ldots, n$*, there are only finitely many* a_0 *for which* $\overline{\mathcal{M}}_{m;n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ *is non-empty .*

3. *It is a manifold of dimension*

$$\begin{aligned}&\dim_{\mathbb{R}} \overline{\mathcal{M}}_{m;n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0) \\ &= (2m + n - 2) + \deg o_{a_0} - \sum_{i=1}^{m} \deg o_{\gamma_i} - \sum_{j=1}^{n} \deg o_{a_j}\end{aligned}$$

Similar result holds for $\overline{\mathcal{M}}_{m,1;n}(\gamma_1, \ldots, \gamma_m, \gamma_0; a_1, \ldots, a_n)$

Proof. See [Gan13]. For a compactness result, one need to assure that the energy of pseudo-holomorphic curves are a priori bounded in M. This estimate is carefully done therein. Transversality result is a standard application of Sard-Smale argument. The dimension formula is also a standard application of Atiyah-Singer index theorem on a linearized Fredholm operator. □

When $\mathcal{M}_{m;n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ has dimension zero so that it is rigid, then a map $u : S \to M$ in that moduli space is isolated. An orientation of the moduli space provides a canonical isomorphism

$$Q_u : \bigotimes_{i=1}^{m} o_{\gamma_i} \otimes \bigotimes_{j=1}^{n} o_{a_j} \to o_{a_0}.$$

We sum up Q_u for all $u \in \mathcal{M}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ and all a_0 and define

$$\mathbf{F}_{m;,n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n)$$

$$:= \sum_{\dim_{\mathbb{R}} \mathcal{M}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n,a_0)=0} \; \sum_{u\in\mathcal{M}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n,a_0)} Q_u\left(\bigotimes_{i=1}^{m} o_{\gamma_i} \otimes \bigotimes_{j=1}^{n} o_{a_j}\right)$$

We define $\mathbf{F}_{m,1;n}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n)$ in a similar way.

2.4 Wrapped Fukaya category

In this section and the next, we recall a definition of wrapped Fukaya category and symplectic cohomology in a quadratic hamiltonian setup . See [Rit13] or [Gan13] for more detailed discussion. For two Lagrangian submanifolds $L_0, L_1 \in \mathcal{W}$, a *wrapped Floer cochain complex* is a vector space

$$CW^\bullet(L_0,L_1;H) = \bigoplus_{a\in\chi(L_0,L_1;H)} o_a$$

It is graded by the degree $\deg o_a$. We will use the notation a instead of o_a for generators if it cause no confusion.

Definition 2.4.1. *A wrapped Fukaya category $\mathcal{W}\mathcal{F}(M)$ consists of*

1. *a set of objects $\mathcal{W}$*
2. *a space of morphisms $CW^\bullet(L_0,L_1)$ for $L_i \in \mathcal{W}$,*
3. *an A_∞ structure*

$$m_k : CW^\bullet(L_0,L_1)\otimes\cdots\otimes CW^\bullet(L_{k-1},L_k) \to CW^\bullet(L_0,L_k)$$
$$m_k(a_1,\ldots,a_k) = (-1)^{\square_k} \boldsymbol{F}_{0;k,1}(a_1,\ldots,a_k);$$
$$\square_k = \sum_1^k i\cdot \deg a_i$$

Recall that $\boldsymbol{F}_{0;k,1}(a_1,\ldots,a_k)$ is given by a counting of a zero-dimensional component of a moduli space of pseudo-holomorphic discs

$$\mathcal{M}_{0;k,1}(a_0;,a_1,\ldots,a_k).$$

The proof of A_∞ relation follows from the degeneration patterns of pseudo-holomorphic discs, which corresponds to a codimension 1 boundary strata of

Gromov bordification $\overline{\mathscr{M}}_{0;k,1}(a_0;,a_1,\ldots,a_k)$. In particular, we have $m_1^2=0$. The cohomology of a complex, denoted by

$$HW^\bullet(L_0,L_1)=H^\bullet\left(CW^\bullet(L_0,L_1;H),m_1\right)$$

, called wrapped Floer cohomology between L_0 and L_1. It does not depends on the choices of hamiltonian H or its perturbation F.

2.5 Symplectic cohomology and closed-open map

Definition 2.5.1. *A symplectic cochain complex is a* $\mathbb{Z}$*-graded cochain complex*

$$CH^\bullet(M;H_{S^1})=\bigoplus_{\gamma\in\mathscr{O}(M;H_{S^1})}o_\gamma$$

graded by the degree $\deg o_\gamma$*. We will use the notation* γ *instead of* o_γ *for generators if it cause no confusion. A differential of this complex is*

$$\mathrm{d}_{CH}(o_{\gamma_1})=(-1)^{\deg o_{\gamma_1}}\boldsymbol{F}_{1,1;0}(\gamma_1).$$

Recall that $\boldsymbol{F}_{1,1;0}(\gamma_1)$ *is given by a counting of a zero-dimensional component of a moduli space of pseudo - holomorphic annulus*

$$\mathscr{M}_{1,1;0}(\gamma_1,\gamma_0).$$

The proof of $\mathrm{d}^2=0$ follows from the degeneration patterns of pseudo-holomorphic annulus, which is a codimension 1 boundary strata of $\overline{\mathscr{M}}_{1,1;0}(\gamma_1,\gamma_0)$. In particular, we have $m_1^2=0$. The cohomology of a complex, denoted by

$$SH^\bullet(M)=H^\bullet\left(CH^\bullet(L_0,L_1;H),\mathrm{d}_{CH}\right)$$

is called symplectic cohomology of M. It is an invariant of M and does not depend on the specific choices of hamiltonians or its perturbation.

A chain level operation

$$CH^\bullet(M)^{\otimes 2} \to CH^\bullet(M) \tag{2.5.1}$$
$$(\gamma_1, \gamma_2) \mapsto (-1)^{\deg \gamma_1} \mathbf{F}_{2,1;0}(\gamma_1, \gamma_2) \tag{2.5.2}$$

induces a ring structure on its cohomology.

Symplectic cohomology ring acts on the wrapped Fukaya category, just like a general ring acts on its modules. Let's start with the definition of Hochschild cohomology of A_∞ category.

Definition 2.5.2. *A closed-open map is a map*

$$\mathrm{CO} : CH^\bullet(M) \to CC^\bullet(\mathscr{W}\mathscr{F}(M), \mathscr{W}\mathscr{F}(M)) \tag{2.5.3}$$
$$CO(\gamma)(a_1, \ldots, a_n) := (-1)^{\square_k} \boldsymbol{F}_{1;n,1}(\gamma; a_1, \ldots, a_n, a_0) \tag{2.5.4}$$
$$\square_k = \sum i \cdot \deg a_k \tag{2.5.5}$$

A degeneration pattern of a moduli space $\overline{\mathscr{M}}_{1;n,1}(\gamma; a_1, \ldots, a_n, a_0)$ proves that CO is a cochain map.

Chapter 3

New A_∞ category $\mathscr{C}_\Gamma$

For a chosen symplectic cohomology class $\Gamma \in SH^0(M)$, we construct a new A_∞ category $\mathscr{C}_\Gamma$ on which the action of Γ vanishes.

3.1 Popsicles with interior markings

Abouzaid-Seidel introduced the notion of a popsicle and popsicle maps to define a homotopy direct limit version of wrapped Fukaya category [AS10]. A popsicle is a punctured disc with interior marked points which can move along special lines on the disc. This interior marked points (called sprinkles) were not used as inputs in [AS10], but as a tracking device to to write various continuation maps (to increase slopes according to weights) in a consistent way. In particular, these marked points were allowed to coincide. Therefore a popsicle with one sprinkle provides continuation maps, which are expected to be isomorphisms in Floer cohomology. Abouzaid-Seidel has described the compactification of moduli of popsicles and the signs for associated A_∞-operations.

We will use a variation of the notion of popsicle, but our usage is completely different from [AS10]. We will use interior marked point as places for actual inputs (given by a symplectic cohomology class). Therefore, the compactification of the moduli space of popsicles is somewhat different from [AS10] in that if interior marked point collide, we introduce sphere bubbles as in the standard Floer theory. Also, we do not use any weights. For example, a popsicle with one sprinkle will be regarded as a quantum cap action of a symplectic cohomology class and the images of quantum cap action will vanish in the new cohomology theory.

A relevant moduli space consists of popsicles.

Definition 3.1.1 (See [AS10]). *A ϕ-flavoured popsicle with interior markings is a disc D^2 with following decorations;*

1. ***boundary markings**: its boundary carries a single outgoing marking denoted by z_0 and n incomming markings denoted by $z_1, \dots, z_n$.*

2. ***popsicle sticks**: geodesic l_i connecting each $z_i (i \geq 1)$ and z_0*

3. ***flavour**: a finite index set F and a set map*

$$\phi : F \to \{1, \dots, n\}.$$

4. ***sprinkles**: a function*

$$x : F \to l_{\phi(f)}$$

such that if $\phi(f_1) = \phi(f_2)$, then $x(f_1) \neq x(f_2)$ for $f_i \in F$.

We called it stable if $n + |F| \geq 2$. We denotes a moduli space of ϕ-flavoured popsicles with n boundary incoming marked points modulo automorphism by $P_{n,F,\phi}$. Also, we denote

$$Sym^\phi \subset S_F$$

a subgroup of a symmetry group S_F which stabilizes ϕ.

Geometrically, a flavour map ϕ and sprinkle x are nothing but an assignment of an interior marked point $x(f)$ on a geodesic $l_{\phi(f)}$. Since we have no conditions on ϕ, two or more interior markings are allowed to be on a same popsicle stick. But because of the additional restriction on x, all interior markings are different from each other. See Figure 3.1. We list some of the basic properties of this moduli space. Since $x(f)$ are points on a infinite geodesic, we can identify them with real numbers. Consider a fiber of the forgetful map

$$\pi : P_{n,F,\phi} \to S_{0;n,1}.$$

When $n \geq 2$, it is a open subset

$$\left\{\vec{x} \in \mathbb{R}^{|F|} \mid x(f_1) \neq x(f_2) \;\text{ when }\; \phi(f_1) = \phi(f_2)\right\}.$$

If we reverse this maps, we get an embedding

$$(\pi, x) : P_{n,F,\phi} \to S_{0;n,1} \times \mathbb{R}^{|F|}.$$

When $n = 1$, then there is only one popsicle stick and a translation of a holomorphic strip becomes an automorphism of a popsicles. Therefore the fiber of a forgetful map is

$$\left\{\vec{x} \in \mathbb{R}^{|F|} \mid x(f_1) \neq x(f_2)\right\} / \mathbb{R}.$$

.t which can be viewed as a subspace of $\mathbb{R}^{|F|-1}$. Notice that Sym^ϕ is trivial if and only if $|F|$ is injective. If not, any transposition of F is an orientation reversing automorphism of $P_{n,F,\phi}$

3.2 Compactification

We keep following [AS10] where the compactification and transversality argument has been established for holomorphic popsicles. But as we remark in the last section, the Gromov bordification $\overline{P}_{n,F,\phi}$ is larger then the original reference. Its boundary strata contains sphere bubbles as depicted in Figure 3.1. Although we won't use that extra component, we include a brief description of it for sake of completeness.

Definition 3.2.1. *A rooted ribbon tree is a tree T with*

- *a root and leaves: $d+1$ semi-infinite edges with a preferred choice of one among them. The preferred one is called the root, and the rest is called leaves.*
- *ribbon structure: a cyclic order on adjacent edges for each vertex v of T.*

The root and leaves determine a direction on edges. Each vertex v has a single adjacent edge e_0 eminating from the root, and the rest are cyclicly ordered as $\{e_1, \dots, e_{val(v)-1}\}$.

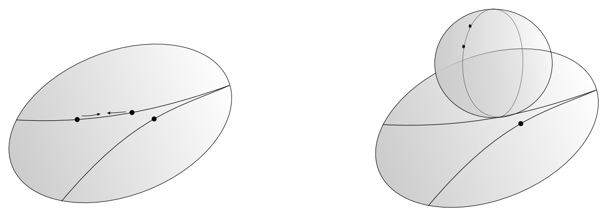


Figure 3.1: (left) Example of $P_{2,F,\phi}$ with $|F| = 3$. (right) A sphere bubble occurs when two or more sprinkles on a same popsicle stick collide

At first, let us describe a model for sphere bubbles.

Definition 3.2.2. *Let T be a rooted tree with no leaves. An F-flavoured icecream modelled on T consists of spheres $\mathbb{P}^1_w$ for each vertex w with the following decorations.*

- *an anti-holomorphic involution $\tau_w : \mathbb{P}^1_w \to \mathbb{P}^1_w$*
- *$val(w)$-special points which is invariant under τ_w and respects a cyclic order at w.*
- *decomposition of a set of flavour $F = \bigsqcup_w F_w$;*
- *a sprinkle function $x_w : F_w \to \mathbb{P}^1_w$ whose image is also τ_w-invariant and disjoint from special points.*

We call F-flavoured icecream is stable if there are more than three special points on each $\mathbb{P}^1_w$.

The reader would immediately notice that a ϕ-flavoured icecream is just a model for a sphere bubble when two or more sprinkles on a same popsicle stick collide. A tree T only determines a configuration of a sphere bubble so it has no leaves. Extra markings other than nodal points are determined from a sprinkle function x_w. An involution τ comes form the following reason; a popsicle stick can be considered as a fixed locus of an anti-holomorphic involution on a disc. Whenever several sprinkles collide, a sphere bubble also carries an involution τ. All nodal points and sprinkles should be τ-invariant.

Definition 3.2.3. *A ϕ-flavoured broken popsicle with icecream on it modeled on a rooted tree T consists of*

- ***decomposition of*** F*: decomposition*

$$F = \bigcup_v F_v$$

and

$$F_v = F'_v \sqcup \bigsqcup_i F'_{v,i}$$

- ***decomposition of*** ϕ*: a map*

$$\phi_v := \phi|_{F_v} : F_v \to \{1, \dots, val(v) - 1\}$$

satisfying the following two conditions.

  1. *for each $f \in F'_v$, the vertex v must lie on the unique path from the root to $e_{\phi_v(f)}$ at v;*
  2. *an image $\phi_v(F'_{v,i})$ is a single point.*

- ***popsicles****: an assignment of ϕ_v-flavoured popsicle on each v such that the sprinkle map x_v is injective on F'_v and constant on $x_v(F'_{v,i})$. Images $x_v(F'_v)$ and $x_v(F'_{v,i})$ are different.*

- ***icecream****: a stable $F'_{v,i}$-flavoured icecream structure modeled on some rooted tree $T'_{v,i}$ with no leaves for each (v, i).*

A ϕ-flavoured broken popsicle with icecream on it is called stable if all popsicles and icecreams are stable.

Although the definition looks complicated, the geometric intuition should be clear. A decomposition of F_v consists of two parts; F'_v is a part on which ϕ is injective, and we assign an ordinary popsicle structures according to its image. On the other hand, $F'_{v,i}$ is a set of sprinkles that collides at the point $\phi(F'_{v,i})$. We attach a sphere bubble, or icecream on that point. Notice that $|F'_{v,i}| \geq 2$ as soon as it is stable.

A moduli space of ϕ-flavoured broken popsicle with icecream modeled on T is a product

$$P^T_{n,F,\phi} = \prod_v P_{val(v)-1,F_v,\phi_v} \times \prod_{T'_{v,i}} \mathbb{R}^{|F'_{v,i}|+|edge(T'_{v,i})|-3|vert(T'_{v,i})|}$$

Take the disjoint union of those spaces, and denote it by

$$\overline{P}_{n,F,\phi} := \bigsqcup_{T,F=\bigsqcup F_v} P^T_{n,F,\phi}.$$

Proposition 3.2.4. *$\overline{P}_{n,F,\phi}$ is a compact smooth manifold with corners.*

Proof. The boundary strata is a mixture of two disjoint degenerations; one is when an underlying disc component breaks into several pieces, and the other is when several sprinkles collide.

The first part can be covered by the result of [AS10]. If we forget about icecream structure and simply allow a sprinkle function x_v may not be injective, then the corresponding moduli is the same as their moduli spaces of ϕ-flavoured popsicles. They construct an algebro-geometric model (called holomorphic lollipops) for such moduli spaces and prove that a standard gluing procedure along strip-like end gives a structure of a smooth manifold with corner on the moduli space.

Then, a second kind of degeneration can be covered easily. This is essentially a compactification of a configuration space of points on S^1. (See Fulton-Macpherson [FM94]). Consider a fiber of a forgetful map

$$\pi_v : P_{val(v)-1,F_v,\phi_v} \to S_{0;val(v)-1,1}$$

It is an open complement of $\mathbb{R}^{|F_v|}$ given by

$$\left\{\vec{x} \in \mathbb{R}^{|F_v|} : x_v(f_1) \neq x_v(f_2), \quad \forall f_i \in F'_{v,i}\right\}$$

A value $x_v(F'_{v,i})$ determines a limit point on a naive compactified fiber $\mathbb{R}^{|F|}$. We perform a consecutive oriented real blow-up on the locus where two or more coordinate coincides until all coordinates are finally distinguished to each other. A rooted tree $T'_{v,i}$ corresponds exactly to a possible boundary strata of this blowups. The number of vertices of $T'_{v,i}$ determines the number of blow-ups you perform to reach that strata. A value of sprinkles x_w then determines a coordinates of a moduli.

A real oriented blow-ups of a smooth compact manifold with corners is again a smooth compact manifold with corners. We finish the proof. □

The structure of a manifold with corners are compatible to a canonical inclusion

$$\mathscr{P}_{n,F,\phi} \subset \overline{S}_{|F|;n,1}.$$

We leave it to an interested reader.

Definition 3.2.5. *Let $a_i \in CF^\bullet(L_{i-1}, L_i)$ and $\Gamma \in SH^0(M)$. Define*

$$\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$$

be a compactified moduli space of pseudo-holomorphic maps

$$\left\{ u : S \to M : S \in \overline{P}_{n,F,\phi} \right\}$$

satisfies

- *a boundary segment from z_{i-1} to z_i goes to L_i,*
- *a boundary marking z_i goes to a_i,*
- *all interior markings are asymptotic to Γ.*

It can be described as a submanifold with corners

$$\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n) \subset \overline{\mathscr{M}}_{|F|;n,1}(\Gamma, \ldots, \Gamma; a_1, \ldots, a_n, a_0)$$

cut out by a popsicle conditions on interior marked points.

A standard compactness and transversality argument now can be applied. Notice that we can choose a Floer data consistently for a family of domains. The moduli space is still a manifold with corners, so we may extend it inductively form the lowest dimensional strata.

Lemma 3.2.6. *For a generic choice of universal and consistent Floer data,*

1. *The moduli spaces* $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ *are smooth and compact.*

2. *For a given input* Γ *and* a_i, $i = 1, \dots, n$, *there are only finitely many* a_0 *for which* $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ *is non-empty .*

3. *It is a manifold of dimension*

$$|F| + n - 2 + \deg a_0 - \sum_{i=1}^{n} \deg a_i$$

Proof. A compactness and transversality argument is mostly the same as before. A standard index formula tells us that

$$\begin{aligned}
\dim_{\mathbb{R}} \overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n) &= \dim_{\mathbb{R}} \overline{\mathscr{M}}_{|F|;n,1}(\Gamma, \dots, \Gamma; a_1, \dots, a_n, a_0) - |F| && (3.2.1)\\
&= (2|F| + n - 2) + \deg a_0 - \sum_{i=1}^{n} \deg a_i - |F| \cdot \deg \Gamma && (3.2.2)\\
&= |F| + n - 2 + \deg a_0 - \sum_{i=1}^{n} \deg a_i. && (3.2.3)
\end{aligned}$$

□

We denote an orientation operator associated to the zero-dimensional component of $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ by

$$m^{\Gamma}_{n,F,\phi}.$$

In particular, $m^{\Gamma}_{n,F} = m_n$ if F is an empty set. A degree of this operator is $2 - n - |F|$

3.2.1 A_∞ category $\mathscr{C}_\Gamma$

In this section, we construct a new A_∞ category $\mathscr{C}_\Gamma$. We start with the following important observation.

Proposition 3.2.7. *If* $\phi : F \to \{1, \dots, n\}$ *is not injective, then* $m^{\Gamma}_{n,F,\phi}$ *vanishes.*

Proof. It means that at least one popsicle stick carries more than two interior markings, which also means ϕ is not injective. Then Sym^{ϕ} contains a nontrivial transposition. Since we put a same class Γ for all interior markings, the transposition extends to $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ also. It induces an orientation-reversal automorphism on $\overline{\mathscr{P}}_{n,F,\phi}$. Therefore the contribution of this moduli space should vanish. □

Now we can focus on the case when $\phi : F \to \{1, \ldots, n\}$ is injective. Then F can be considered as a subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$. In this case, we omit a notation ϕ and simply write $\overline{\mathscr{P}}_{n,F}$ and $m^\Gamma_{n,F}$.

Definition 3.2.8. *An admissible cut of F consists of*

1. $n_1, n_2 \geq 1$ *such that* $n_1 + n_2 = n + 1$
2. *a number* $i \in \{1, \ldots, n\}$
3. $F_1 \subset \{1, \ldots, n_1\}$ *and* $F_2 \subset \{1, \ldots, n_2\}$ *such that* $|F_1| + |F_2| = |F|$

satisfies the following property;

- $F \supset \{k | k \in F_1, k < i\}$ *and* $F \supset \{k + n_2 - 1 | k \in F_1, k > i\}$
- $F \supset \{k + i - 1 | k \in F_2\}$

If $i \notin F_1$*, then this completely recovers* F*. Otherwise,* F *has one more element among* $\{i, i + 1, \ldots, (i + n_2 - 1)\}$.

An admissible cut describes a stratum $P_{n_1,F_1} \times P_{n_2,F_2}$ of a moduli space $\overline{P}_{n,F}$. They describes precisely codimension 1 strata whose associated sprinkle $\phi_j : F_j \to \{1, \ldots, n_j\}$ $(j = 1, 2)$ is still injective. Combined with 3.2.7, we get a quadratic relation

$$\sum_{\forall \text{admissible cuts}} (-1)^{\square} m^\Gamma_{n_1,F_1}(a_1, \ldots, a_{i-1}, m_{n_2,F_2}(a_i, \ldots, a_{i+|F_2|}), a_{i+1+|F_2|}, \ldots, a_{n_1+n_2-1})$$
$$= 0$$

Now we are ready to define a new A_∞ category.

Definition 3.2.9. *Let* $\Gamma \in SH^0(M)$*. A category* $\mathscr{C}_\Gamma$ *consists of*

1. *a set of objects* $Ob(\mathscr{C}_\Gamma) = Ob(\mathscr{W}\mathscr{F}(M))$.

2. *morphisms between two objects*

$$\mathrm{Hom}_{\mathscr{C}_\Gamma}(L_1, L_2) = CW(L_1, L_2) \oplus CW(L_1, L_2)[1]$$

We denotes the element of this complex by $c := a + \epsilon b$ with $deg \epsilon = -1$

3. *An A_∞ structure $\{M_n\}_{n=1}^{\infty}$ is given as follows. We may write*

$$M_n(c_1, \cdots, c_n) = M_n^a(c_1, \cdots, c_n) + \epsilon M_n^b(c_1, \cdots, c_n)$$

   (a) *Suppose $c_i = a_i$ for all i (all the inputs do not have ϵ components), then we set*

$$M_n(a_1, \cdots, a_n) = m_n(a_1, \cdots, a_n)$$

   where $\{m_n\}$ is the A_∞-operation for $\mathscr{W}\mathscr{F}(M)$.

   (b) *Suppose $c_i = \epsilon b_i$ for $i \in \{i_1, \ldots, i_k\}$, and $c_i = a_i$ for $i \notin \{i_1, \ldots, i_k\}$. Then we set*

$$F = \{i_1, \ldots i_k\}, \quad \widehat{F}^j = \{i_1, \ldots, \widehat{i_j}, \ldots, i_k\},$$

   and define

$$M_n(c_1, \ldots, c_n) = M_n^a(c_1, \ldots, c_n) + \epsilon M_n^b(c_1, \ldots, c_n)$$

$$M_n^a(c_1, \ldots, c_n) = (-1)^{\epsilon_a} m_{n,F}^{\Gamma}(a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, b_{i_k}, \ldots, a_n)$$

$$M_n^b(c_1, \ldots, c_n) = \sum_{j=1}^{k} (-1)^{\epsilon_j + \epsilon_{b,j}} m_{n,\widehat{F}^j}^{\Gamma}(a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, b_{i_k}, \ldots, a_n)$$

   If we use the notion x_i to denote b_i for $i \in F$ and a_i for $i \notin F$. Denote by $|x|' = |x| - 1$. Then

$$\begin{aligned}
\epsilon_a &= \sum_j |x_j| + \sum_{f \in F, l > f} (|x_l| - 1) \\
\epsilon_{b,j} &= \sum_j |x_j| + \sum_{f \in \widehat{F}^j, l > j} |x_l|' \\
\epsilon_j &= \sum_{l=1}^{j-1} |x_l|'
\end{aligned}$$

Remark 3.2.10. *Although geometric setting of this A_∞-category and that of Abouzaid-Seidel [AS10] (homotopy limit chain complex of wrapped Fukaya category) are completely different, the resulting algebraic structures has strong similarities because both are based on some versions of "popsicle" moduli spaces. In particular, we can use the sign analysis of popsicle moduli space of [AS10] to have the same sign as above.*

As a sanity check, we check the degree of A_∞-operations. Recall the degree of $m^\Gamma_{n,F}$ is $2-n-|F|$. Additional degree shift $-|F|$ comes from interior markings. We correspondingly shift our inputs b_i for each interior markings by multiplying ϵ. Therefore, a degree of M_n becomes $2-n$.

Proposition 3.2.11. *$\mathscr{C}_\Gamma$ is an A_∞ category. Namely, for any composable $(c_1, \cdots, c_n)$, we have*

$$\sum_{n_1+n_2=n+1} (-1)^{\sum_{j=1}^{i-1}|c_j|'} M_{n_1}(c_1, \cdots, c_{i-1}, M_{n_2}(c_i, \cdots, c_{i+n_2-1}), \cdots, c_n) = 0$$

Proof. We check the identity on each component of the output. We first show that

$$\sum \left(M^a_{n_1}(\cdots, M^a_{n_2}(\cdots), \cdots) + M^a_{n_1}(\cdots, M^b_{n_2}(\cdots), \cdots) \right) = 0.$$

This identify follows from the compactification of popsicle moduli space. Namely, A codimension one statra of popsicle moduli space corresponds to a term in the above equation. In Figure 3.2, we illustrated corresponding broken popsicles in the same order for the case $|F| = 4$.

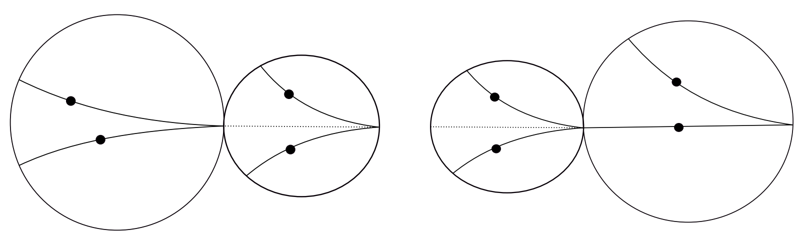


Figure 3.2: A_∞-identity with a-output

Next we show that

$$\sum \left(M^b_{n_1}(\cdots, M^a_{n_2}(\cdots), \cdots) + M^b_{n_1}(\cdots, M^b_{n_2}(\cdots), \cdots) \right) = 0.$$

This identify follows from the compactification of popsicle moduli space for $\widehat{F}^j$ for all j. In Figure 3.3, we illustrated corresponding broken popsicles in the same order for the case $|F| = 4$ and $j = 1$.

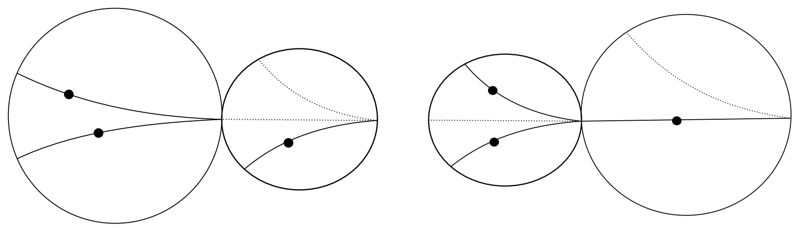


Figure 3.3: A_∞-identity with b-output

Now let us explain the signs. We will be very brief, since how to construct orientation of Fukaya category is by now well-understood. Also, Abouzaid-Seidel already carried out detailed sign analysis for popsicles and popsicle maps, which we can easily adapt to our setting. In particular note that the signs appearing in the definition of A_∞-operation here are the same as Section 3h [AS10].

First, recall that we are using the sprinkle as a place for interior Γ-insertion whereas in [AS10] sprinkles are just a marker for some other data. Hence orientation for the latter for a sprinkle is given by the orientation of $\mathbb{R}$ (the popsicle stick), but in our case, we need $\mathbb{R} \otimes o_\Gamma$ where o_Γ is the orientation operator of the Reeb orbit Γ. It is important that our symplectic cohomology insertion Γ has even degree so that it does not affect any sign for switching places. We refer readers to [AS10] for detailed explanation for signs, and leave the adaptation as an exercise. $\square$

3.3 Cohomology category

In this subsection, we describe $\mathscr{C}_\Gamma$ at its cohomology level. A differential is given by

$$M_1(a+\epsilon b) = \left(m_1(a) + m^\Gamma_{1,\{1\}}(b)\right) + \epsilon m_1(b), \quad a, b \in CW(L_1, L_2).$$

Proposition 3.3.1. *Up to homotopy, we have*

$$m^\Gamma_{1,1}(b) = m_2\left(b, \mathrm{CO}_{L_2}(\Gamma)\right)$$

where $CO_{L_2} : SH^\bullet(M) \to CW^\bullet(L_2, L_2)$. is a word-length zero component of the closed-open map.

Proof. Consider the moduli space of popsicles $P_{1,\{1\}}$. The moduli space is isolated. It is a moduli space of disc with one outgoing boundary marking $z_0 = 1$, one incoming boundary marking $z_1 = -1$ and also a single interior marking $x_0 = 0$. Now consider a 1-parameter family of moduli space of holomorphic discs with

- one outgoing boundary marking z_0 at 1
- one incoming boundary marking z_1 at -1
- one moving interior marking x_0 at $-it, \quad t \in [0,1]$

At $t = 0$, we get $P_{1,\{1\}}$. At $t = 1$, we get a moduli space of discs with disc bubbles containing interior marked points. See Figure 3.4. It corresponds to a disc moduli space governing $m_2\left(b, \mathrm{CO}_{L_2}(\Gamma)\right)$. $\square$

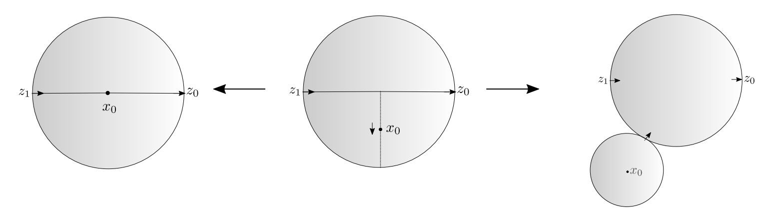


Figure 3.4: Homotopy between $m^{\Gamma}_{1,\{1\}}$ and $\mathrm{CO}(\Gamma)$

Corollary 3.3.2. *As a complex,*

$$\mathrm{Hom}^\bullet_{\mathscr{C}_\Gamma}(L_1, L_2) \simeq \mathrm{Cone}\left(CW^\bullet(L_1, L_2) \xrightarrow{m_2(-.\mathrm{CO}_{L_2}(\Gamma))} CW^\bullet(L_1, L_2) \right)$$

Therefore a category $\mathscr{C}_\Gamma$ is an A_∞ category on which an action of Γ by a closed-open map vanishes homotopically. This observation becomes even more clear if we formulate it in the language of bimodules. Let's restrict our input to be of the form

$$(a_1 \otimes \cdots \otimes a_i \otimes \underline{b} \otimes a_{i+1} \otimes \cdots \otimes a_n) \in (\mathscr{WF}^{\otimes i}) \otimes \underline{\mathscr{WF}} \otimes (\mathscr{WF}^{\otimes n-i})$$

Here we underline the middle component to emphasize that we consider ϵb as an element of bimodules.

Definition 3.3.3 (Quantum cap action of Γ)**.** *A cocain level Quantum cap action of Γ is an A_∞ pre-bimodule map defined by*

$$\cap\Gamma : T(\mathscr{WF}) \otimes \underline{\mathscr{WF}} \otimes T(\mathscr{WF}) \to \mathscr{WF} \tag{3.3.1}$$

$$(a_1, \ldots, a_i, \underline{b}, a_{i+1}, \ldots, a_n) \mapsto M^a_{k+1}(a_1, \ldots, a_i, \epsilon b, a_{i+1}, \ldots, a_n). \tag{3.3.2}$$

Indeed, it is a bimodule homomorphism from $\mathscr{WF}(M)$ to itself. We only have to show that the differential of this pre-bimodule map vanishes, which is just a part of **??** when $|F| = 1$. Therefore we found a distinguished triangle

$$\mathscr{WF}(M) \xrightarrow{\cap\Gamma} \mathscr{WF}(M) \longrightarrow \mathscr{C}_\Gamma \longrightarrow$$

of bimodules.

Remark 3.3.4. *A cochain level $\cap\Gamma$ descend to a cohomology category $H(\mathscr{WF}(M))$, which is the standard quantum cap action as explained in [Aur07].*

On the other hand, a complex of bimodule homomorphism $\mathrm{Hom}_{\mathscr{A}-\mathscr{A}}(\Delta_{\mathscr{A}}, \Delta_{\mathscr{A}})$ is one of the presentation of Hochschild cohomology $HH^\bullet(\mathscr{A}, \mathscr{A})$. In [Gan13], a bimodule version of closed-open map, called two-pointed open-closed map,

$$^2\mathrm{CO} : SH^\bullet(M) \to {}^2CC^\bullet(\mathscr{WF}(M), \mathscr{WF}(M))$$

was constructed. One can check directly that the quantum cap action $\cap\Gamma$ coincides to $^2\mathrm{CO}(\Gamma)$

Intuitively,

- objects of $\mathscr{C}_\Gamma$ are a twisted complex

$$\left(L \xrightarrow{\mathrm{CO}(\Gamma)} L \right), \quad L \in \mathscr{WF}(M).$$

In many cases, they can be realized as a geometric surgery of L with itself along $\mathrm{CO}(\Gamma)$.

- the space of morphisms is a "half" of the original one. It consists of

$$"a+\epsilon b" = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in CW^\bullet\left(L_1 \xrightarrow{\mathrm{CO}(\Gamma)} L_1 \ , \ L_2 \xrightarrow{\mathrm{CO}(\Gamma)} L_2 \right)$$

Of course, this intuitive analogy is simply not true in all possible ways. Let's use more than one interior marking. Unlike $\widetilde{m}_1$, $\widetilde{m}_2$ operation is already counter-intuitive.

$$m_2(\epsilon b_1, \epsilon b_2) = m^\Gamma_{2,\{1,2\}}(b_1, b_2) + \epsilon(\text{extra term}).$$

One might expect $\widetilde{m}_2(\epsilon b_1, \epsilon b_2)$ vanishes, according to the intuition. It is not true. In fact, we will exibit an example that $\widetilde{m}_2(\epsilon b_1, \epsilon b_2) = 1$ at the level of cohomology.

3.4 Example: M_2-operation

Let us examine the Leibniz rule for the input $(a, \epsilon b)$. Namely, we want to verify

$$M_1(M_2(a,\epsilon b)) + M_2(M_1(a), \epsilon b) + (-1)^{|a|'} M_2(a, M_1(\epsilon b)) = 0. \tag{3.4.1}$$

From the definition

$$M_2(a, \epsilon b) = m^\Gamma_{2,\{2\}}(a,b) + \epsilon m_2(a,b)$$

$$M_1(\epsilon b) = m^\Gamma_{1,\{1\}}(b) + \epsilon m_1(b)$$

We have

$$\begin{aligned} M_1(M_2(a,\epsilon b)) &= M_1(m^\Gamma_{2,\{2\}}(a,b) + \epsilon m_2(a,b)) \\ &\quad = m_1(m^\Gamma_{2,\{2\}}(a,b)) + \left(m^\Gamma_{1,\{1\}}(m_2(a,b)) + \epsilon m_1(m_2(a,b))\right) \\ M_2(M_1(a), \epsilon b) &= M_2(m_1(a), \epsilon b) = m^\Gamma_{2,\{2\}}(m_1(a), b) + \epsilon m_2(m_1(a), b) \\ M_2(a, M_1(\epsilon b)) &= M_2(a, m^\Gamma_{1,\{1\}}(b) + \epsilon m_1(b)) \\ &= m_2(a, m^\Gamma_{1,\{1\}}(b)) + m^\Gamma_{2,\{2\}}(a, m_1(b)) + \epsilon M_2(a, m_1(b)) \end{aligned}$$

If we collect the terms with ϵ in (3.4.1), we obtain the original A_∞-identity

$$\epsilon\big(m_1(m_2(a,b)) + m_2(m_1(a),b) + (-1)^{|a|'} m_2(a, m_1(b))\big) = 0.$$

Collecting the terms without ϵ in (3.4.1), we get the following

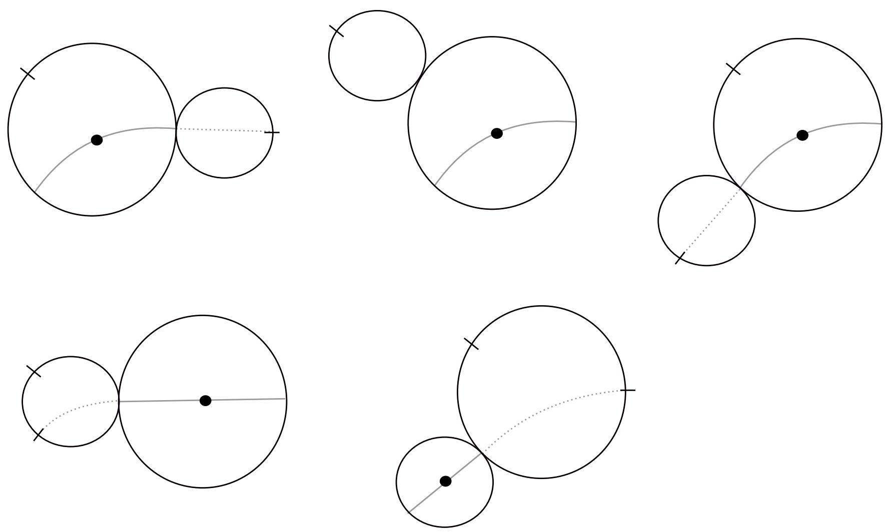


Figure 3.5: Leibniz rule for the inputs $(a, \epsilon b)$

$$m_1 m^{\Gamma}_{2,\{2\}}(a,b) + m^{\Gamma}_{2,\{2\}}(m_1(a),b) + m^{\Gamma}_{2,\{2\}}(a, m_1(b))$$

$$+ m^{\Gamma}_{1,\{1\}}(m_2(a,b)) + m_2(a, m^{\Gamma}_{1,\{1\}}(b)) = 0.$$

These terms correspond to the codimension one degenerations (given by disc bubblings) in Figure 3.5. Here dotted lines just indicate paths to the 0-th vertex, and do not give any restriction to the domain. Hence one may remove dotted lines to find the corresponding A_∞-operations.

Let us examine Leibniz rule for the input $(\epsilon b_1, \epsilon b_2)$. Namely, we want to verify

$$M_1(M_2(\epsilon b_1, \epsilon b_2)) + M_2(M_1(\epsilon b_1), \epsilon b_2) + (-1)^{|b_1|} M_2(\epsilon b_1, M_1(\epsilon b_2)) = 0. \tag{3.4.2}$$

We have

$$\begin{aligned}
M_1(M_2(\epsilon b_1,\epsilon b_2)) &= M_1(m^\Gamma_{2,\{1,2\}}(b_1,b_2)+\epsilon m^\Gamma_{2,\{2\}}(b_1,b_2)+\epsilon m^\Gamma_{2,\{1\}}(b_1,b_2)) \\
&= m_1(m^\Gamma_{2,\{1,2\}}(b_1,b_2)) + m^\Gamma_{1,\{1\}}(m^\Gamma_{2,\{2\}}(b_1,b_2)+m^\Gamma_{2,\{1\}}(b_1,b_2)) \\
&\quad +\epsilon m_1\big(m^\Gamma_{2,\{2\}}(b_1,b_2)+m^\Gamma_{2,\{1\}}(b_1,b_2)\big) \\
M_2(M_1(\epsilon b_1),\epsilon b_2) &= M_2(m^\Gamma_{1,\{1\}}(b_1)+\epsilon m_1(b_1),\epsilon b_2) \\
&= m^\Gamma_{2,\{2\}}(m^\Gamma_{1,\{1\}}(b_1),b_2)+\epsilon m_2(m^\Gamma_{1,\{1\}}(b_1),b_2) \\
&\quad +m^\Gamma_{2,\{1,2\}}(m_1(b_1),b_2)+\epsilon m^\Gamma_{2,\{2\}}(m_1(b_1),b_2)+\epsilon m^\Gamma_{2,\{1\}}(m_1(b_1),b_2) \\
M_2(\epsilon b_1, M_1(\epsilon b_2)) &= M_2(\epsilon b_1, m^\Gamma_{1,\{1\}}(b_2)+\epsilon m_1(b_2)) \\
&= m^\Gamma_{2,\{1\}}(b_1, m^\Gamma_{1,\{1\}}(b_2))+\epsilon m_2(b_1, m^\Gamma_{1,\{1\}}(b_2)) \\
&\quad +m^\Gamma_{2,\{1,2\}}(b_1,m_1(b_2))+\epsilon m^\Gamma_{2,\{2\}}(b_1,m_1(b_2))+\epsilon m^\Gamma_{2,\{1\}}(b_1,m_1(b_2))
\end{aligned}$$

The following figure 3.6 describes the terms without ϵ in the above (in the same order). It is not hard to see that these arise from codimension boundary

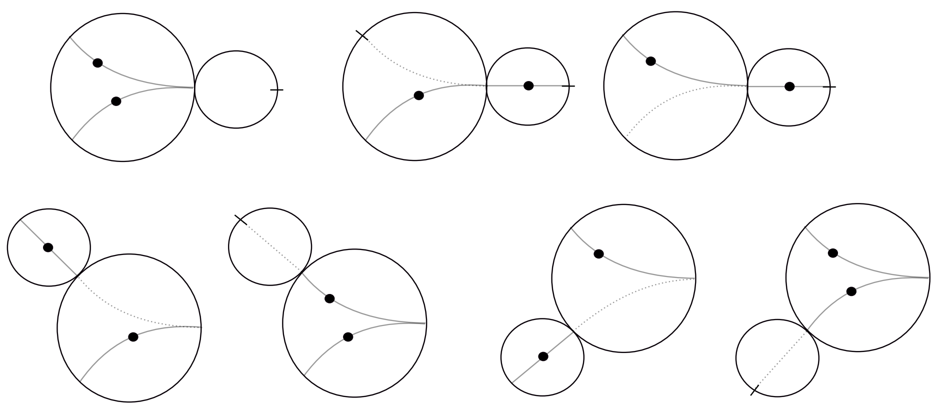


Figure 3.6: Leibniz rule for the inputs $(\epsilon b_1,\epsilon b_2)$

strata of $\overline{P}_{2,\{1,2\}}$. This The terms with ϵ are similar. As we can see from this example, there is no reason to expect $M_1\circ M_2(\epsilon a,\epsilon b)=0$. This make $\{M_k\}$ a bit counter-intuitive.

Chapter 4

Algebro-geometric counterpart

We discuss classical algebro-geometric operation of restricting to a hypersurface in D^bCoh and MF. We will later show that monodromy of wrapped Fukaya category of a Milnor fiber is mirror to this restriction operation.

4.1 Restricting to a hypersurface in D^bCoh

Let S be an algebra. Choose an element

$$g \in Z(S) \cong HH^0(S,S)$$

The bimodule $S \xrightarrow{g} S$ is quasi-isomorphic to an ideal quotient $S/(g)$, and it carries a natural algebra structure. One can directly construct DG algebra structure on the bimodule itself:

Definition 4.1.1. *Define a DG algebra*

$$\mathscr{B} := S[\epsilon] \Big/ \begin{pmatrix} \epsilon^2 = 0 \\ d\epsilon = g \end{pmatrix}, \quad deg\,\epsilon = -1$$

Here the differential d on S is set to be zero.

One can check that multiplication is well-defined.

Further assume that S is commutative. Consider an affine variety $X = \mathrm{Spec}(S)$ and a hypersurface $Y = V(g)$ with an inclusion $i : Y \hookrightarrow X$. We have the following elementary lemma whose proof is omitted.

Lemma 4.1.2. *We have an equivalence* $\mathscr{B} \simeq i_*\mathscr{O}_Y$*. Moreover, we have the following.*

1. *A sheaf* $\mathscr{F}$ *on a hypersurface* Y *corresponds to an* $\mathscr{B}$*-module object. It is a pair* $(i_*\mathscr{F}, h_{\mathscr{F}})$ *where* $i_*\mathscr{F}$ *is a pushfoward of* $\mathscr{F}$ *equipped with a homotopy* $h_{\mathscr{F}}$ *between the zero map and a multiplication of* g*. It is an action of* $\epsilon \in \mathscr{B}$*.*

2. *Moreover,*

$$Hom_Y(\mathscr{F}_1, \mathscr{F}_2) \simeq Hom_{\mathscr{B}}((i_*\mathscr{F}_1, h_{\mathscr{F}_1}), (i_*\mathscr{F}_2, h_{\mathscr{F}_2})).$$

For the sheaf $\mathscr{O}_Y$ on Y, its pushfoward $i_*\mathscr{O}_Y$ has a simple free resolution.

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{g} \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y \longrightarrow 0$$

An action of degree -1 element ϵ, or a homotopy h, is given as follows.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X \arrow[r, "g"] \arrow[dl, "0"'] & \mathscr{O}_X \arrow[r] \arrow[dl, "id"'] & 0 \arrow[dl, "0"'] \\
0 \arrow[r] & \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \arrow[r] & 0
\end{tikzcd}$$

We can recover a category of coherent sheaves on Y in terms of X.

Theorem 4.1.3. *(See [Pre11]) Let* $Y \subset X$ *as before. Then*

$$DCoh(Y) \simeq \mathscr{B} - mod(DCoh(X))$$

Proof. This is a standard application of Lurie's Barr-Beck-theorem. We present an elementary proof to illustrate the idea. Since everything is affine, It is enough to consider a structure sheaf $\mathscr{O}_Y \in DCoh(Y)$. Computation shows that the morphism complex is

$$Hom^{-1}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h)) \simeq \left\{ \begin{tikzcd} \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \arrow[dl, "a_{21}"'] \\ \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \end{tikzcd} \;\middle|\; a_{21} \text{ can be arbitrary} \right\} \tag{4.1.1}$$

$$Hom^{0}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h)) \simeq \left\{ \begin{tikzcd} \mathscr{O}_X \arrow[r, "g"] \arrow[d, "a_{11}"] & \mathscr{O}_X \arrow[d, "a_{22}"] \\ \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \end{tikzcd} \;\middle|\; a_{11} = a_{22} \right\} \tag{4.1.2}$$

$$Hom^{1}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h)) \simeq \left\{ \begin{tikzcd} \mathscr{O}_X \arrow[r, "g"] \arrow[dr, "a_{12}"] & \mathscr{O}_X \\ \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \end{tikzcd} \;\middle|\; h \circ a_{12} = 0 \text{ implies } a_{12} = 0 \right\} \tag{4.1.3}$$

Therefore, $Hom^{\bullet}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h))$ is isomorphic to

$$H^{\bullet}(\ 0 \longrightarrow Hom_X(\mathscr{O}_X,\mathscr{O}_X) \xrightarrow{d=g} Hom_X(\mathscr{O}_X,\mathscr{O}_X) \longrightarrow 0\ ) \simeq Hom^{\bullet}_Y(\mathscr{O}_Y,\mathscr{O}_Y).$$

□

Intuitively, objects of $\mathscr{B} - mod(DCoh(X))$ are cones

$$\left(\ \mathscr{F}[1] \xrightarrow{g} \mathscr{F}\ \right),\ \ \mathscr{F} \in DCoh(X).$$

It is quasi-isomorphic to a quotient $\mathscr{F}/(g)$. And the space of morphisms is a half of the original one. It consists of

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in \mathrm{Hom}^{\bullet}_X\left(\ \mathscr{F}_1[1] \xrightarrow{g} \mathscr{F}_1\ ,\ \mathscr{F}_2[1] \xrightarrow{g} \mathscr{F}_2\ \right).$$

Indeed, this is closed under DG operations.

4.2 Restricting to a graph hypersurface in Matrix factorizations

Consider a non-isolated singularity of the form

$$U = U_1(x_1,\dots,x_{n-1}) + x_n \cdot U_2(x_1,\dots,x_{n-1}).$$

We consider a graph of some polynomial g

$$\begin{aligned} \mathbb{A}^{n-1} &\to \mathbb{A}^n \\ (x_1,\dots,x_{n-1}) &\mapsto (x_1,\dots,x_{n-1},g) \end{aligned}$$

and a pull-back

$$V(x_1,\dots,x_{n-1}) = U(x_1,\dots,x_{n-1},g)$$

of U along this graph. We assume V is an isolated singularities. We have a relation

$$U = V + (x_n - g)U_2$$

inside We explain how to obtain a similar relation between $\mathrm{MF}(U + x_n \cdot V)$ and $\mathrm{MF}(U)$. We start by collecting functorial properties between two matrix factorization categories, which we refer to [Orl09] and [Pos11].

Let $X = \{U = 0\} \subset \mathbb{C}^n$ and $Y = \{V = 0\} \subset \mathbb{C}^{n-1}$. We view Y as a hypersurface $\{g = x_n\} \subset X$. A closed embedding $Y \hookrightarrow X$ is proper and has a finite tor-dimension. A usual adjoint pair of functors (i^*, i_*) extends to categories of singularities.

$$i^* : D^b_{sg}(X) \longleftrightarrow D^b_{sg}(Y) : i_*$$

On the other hand, there is Orlov's equivalences

$$\mathrm{MF}(U) \simeq \overline{D^b_{sg}(X)}, \ \ \mathrm{MF}(V) \simeq \overline{D^b_{sg}(Y)}$$

Here, $\overline{C}$ denotes Karoubi completion of a category C. This functor sends

$$M = \Big(\ M^{odd} \underset{\phi_{01}}{\overset{\phi_{10}}{\rightleftarrows}} M^{even}\ \Big) \mapsto \mathrm{coker}(\phi_{10})$$

We have an induced pair

$$i^* : \mathrm{MF}(U) \longleftrightarrow \mathrm{MF}(V) : i_*$$

Proposition 4.2.1. *Let*

$$M = \left(M^{odd} \underset{\phi_{01}}{\overset{\phi_{10}}{\rightleftarrows}} M^{even} \right) \in \mathrm{MF}(U),$$

$$N = \left(N^{odd} \underset{\psi_{01}}{\overset{\psi_{10}}{\rightleftarrows}} N^{even} \right) \in \mathrm{MF}(V)$$

Then

1. (i^*, i_*) *is an adjoint pair.*
2. $i^* M \simeq M|_{x_n=g} \in \mathrm{MF}(V)$.
3. $i_* N \simeq N \otimes \left(\begin{tikzcd} \mathbb{C}[x_1,\ldots,x_n] \arrow[r, bend left, "x_n-g"] & \mathbb{C}[x_1,\ldots,x_n] \arrow[l, bend left, "U_2"] \end{tikzcd} \right) \in \mathrm{MF}(U)$
4. $(i_* \circ i^*)M = \mathrm{Cone}((x_n - g) : M[1] \to M) \in \mathrm{MF}(U)$

Proof. The first proposition is proven in more general setup. See [Pos11] Section 2.1. Second proposition follows from the fact that cokernel commutes with tensor product.

$$Coker(\phi_{10}) \otimes_{\mathbb{C}[x_1,\ldots,x_n]} \mathbb{C}[x_1,\ldots,x_{n-1}] \simeq \mathrm{coker}(\phi_{10}|_{x_n=g}).$$

To prove a third proposition, we should specify Fourier-Mukai kernel of a push-forward functor. Write

$$V(x_1,\ldots,x_{n-1}) - V(y_1,\ldots,y_{n-1}) = \sum_{i}^{n-1} (x_i - y_i) \cdot V_i$$

Define a Koszul-type matrix factorization Γ of

$$V(\vec{x}) - U(\vec{y}) = V(\vec{x}) - \big(V(\vec{y}) + (y_n - g)U_2(\vec{y})\big)$$

as

$$\Gamma := \left(\Lambda^\bullet\langle e_1,\dots,e_n\rangle, \left(\sum_1^{n-1}(x_i-y_i)i_{e_i} + (y_n-g)i_{e_n} + \sum_1^{n-1} V_i(\cdot\wedge e_i) + U_2(\cdot\wedge V)\right)\right).$$

Under Orlov's equivalence Γ corresponds to a stabilization of a graph $\Gamma_{Y\to X}$. Therefore a Fourier-Mukai functor associated to Γ is a pushforward functor. Notice that

$$-\otimes\Gamma \simeq -\otimes\Delta_V \otimes \left(\begin{tikzcd} \mathbb{C}[x_1,\dots,x_n] \arrow[r, bend left, "x_n-g"] & \mathbb{C}[x_1,\dots,x_n] \arrow[l, bend left, "U_2"] \end{tikzcd}\right).$$

where Δ_V is a stabilized diagonal of V. This proves the third proposition.

For the fourth proposition, observe that $i_*\circ i^*M$ goes to

$$\operatorname{coker}\left(\phi_{10}|_{x_n=g} : M^{odd}|_{x_n=g} \to M^{even}|_{x_n=g}\right)$$

under Orlov's equivalence. It is easy to see that the periodic tail of the following double complex realizes the matrix factorization associated to that module.

$$\begin{tikzcd} \cdots \arrow[r, "\phi_{10}"] & M^{even} \arrow[r, "\phi_{01}"] \arrow[d, "x_n-g"] & M^{odd} \arrow[r, "\phi_{10}"] \arrow[d, "x_n-g"] & M^{even} \arrow[d, "x_n-g"] \\ \cdots \arrow[r, "\phi_{10}"] & M^{even} \arrow[r, "\phi_{01}"] & M^{odd} \arrow[r, "\phi_{11}"] & M^{even} \end{tikzcd}$$

This is equal to $\operatorname{Cone}((x_n-g) : M[-1]\to M)$. □

An analogy of a function ring for matrix factorization is its Jacobian ring $\operatorname{Jac}(U)$ and its element acts on a matrix factorization by a multiplication. We get another DG model for $\operatorname{MF}(V)$ as an object inside $\operatorname{MF}(U)$ with vanishing (x_n-g)-action.

Corollary 4.2.2. *Define a DG algebra*

$$\mathscr{B} := \operatorname{Jac}(U)[\epsilon]\Big/\begin{pmatrix} \epsilon^2 = 0 \\ d\epsilon = x_n - g \end{pmatrix}, \quad deg\epsilon = -1$$

Then

$$\mathrm{MF}(V) \simeq \mathscr{B} - mod\,(\mathrm{MF}(U))$$

Proof. Again, this should be a corollary of Barr-Beck-Lurie theorem. In detail, since $U_2 = \partial_{x_n} U$, a multiplication of U_2 is homotopically zero. We see

$$i_* N \simeq \left(\; M[1] \xrightarrow{x_n - g} M \; \right) \simeq M[\epsilon] \Big/ \left(\begin{array}{l} \epsilon^2 = 0 \\ d\epsilon = x_n - g \end{array} \right)$$

for some $M \in \mathrm{MF}(U)$ satisfying $I^* M = N$. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{B}}(i_* N_1, i_* N_2) &\simeq \mathrm{Hom}_{\mathscr{B}}(M_1[\epsilon], M_2[\epsilon]) && (4.2.1)\\
&\simeq \mathrm{Hom}_{\mathrm{MF}(U)}(M_1, M_2[\epsilon]) \Big/ \left(\begin{array}{l} \epsilon^2 = 0 \\ d\epsilon = x_n - g \end{array} \right) && (4.2.2)\\
&\simeq \mathrm{Hom}_{\mathrm{MF}(U)}(M_1, (i_* \circ i^*) M_2) && (4.2.3)\\
&\simeq \mathrm{Hom}_{\mathrm{MF}(V)}(N_1, N_2). && (4.2.4)
\end{aligned}$$

□

Chapter 5

Equivariant topology of Milnor fiber for invertible curve singularities

In this section, we explain the topology of an invertible curve singularity W. Namely we first describe topology of its Milnor fiber $M_W = W^{-1}(1)$ and their maximal symmetry group G_W. We show in Proposition 5.1.5 that the quotient $[M_W/G_W]$ is homeomorphic to an orbifold sphere with three special points, which are either orbifold points or (orbifold) punctures. We also describe orbifold covering and an action of a deck transformation group in detail.

5.1 Topology of a Milnor fiber

This chapter is mostly borrowed form [Jeo19]. Recall that Milnor fiber is homotopy equivalent to the bouquet of μ-circles where μ is the Milnor number of the singularity.

Lemma 5.1.1. *The weights and Milnor numbers of curve singularities are as follows.*

1. *Weights of* $F_{p,q}$ *are* $(q, p; pq)$. *Its Milnor number is* $(p-1)(q-1)$.
2. *Weights of* $C_{p,q}$ *are* $(q, p-1; pq)$. *Its Milnor number is* $pq - p + 1$.
3. *Weights of* $L_{p,q}$ *are* $(q-1, p-1; pq-1)$. *Its Milnor number is* pq.

Proof. Milnor numbers can be easily computed by the following formula.

Theorem 5.1.2 (Milnor-Orlik)**.** *[MO70] Let $f(x_1, \cdots, x_{n+1})$ be the weighted homogeneous polynomial of weights $(w_1, \cdots, w_{n+1}, h)$. Then, it has isolated singularity at the origin whose Milnor number is given by*

$$\mu(0) = (\frac{h}{w_1} - 1) \cdots (\frac{h}{w_{n+1}} - 1)$$

□

Definition 5.1.3. *The maximal diagonal symmetry group G_W of W is defined by*

$$G_W = \left\{(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2 \mid W(\lambda_1 x, \lambda_2 y) = W(x, y)\right\}$$

It is easy to check the following.

Lemma 5.1.4. *$G_{F_{p,q}} \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$, $G_{C_{p,q}} \simeq \mathbb{Z}/pq\mathbb{Z}$ and $G_{L_{p,q}} \simeq \mathbb{Z}/(pq-1)\mathbb{Z}$.*

Proof. $G_{F_{p,q}} = \left\{\left(\exp\left(\frac{2k\pi\sqrt{-1}}{p}\right), \exp\left(\frac{2l\pi\sqrt{-1}}{q}\right)\right) \middle| \, 0 \le k \le p-1, 0 \le l \le q-1\right\}$. The generators of $G_{C_{p,q}}$ and $G_{L_{p,q}}$ are (ξ^{-p}, ξ) and (η, η^{-p}) respectively for $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{pq}\right)$, $\eta = \exp\left(\frac{2\pi\sqrt{-1}}{pq-1}\right)$. □

For curve singularities, M_W is given by a (non-compact) Riemann surface. Recall that the boundary of a Milnor fiber is called the link of the singularity which are union of k circles for curve singularities. In our case, we compactify M_W to $\overline{M}_W$ by shrinking each circle of the link to a point. Therefore, $\overline{M}_W \setminus M_W$ consists of k-points. G_W acts on M_W as well as $\overline{M}_W$.

Proposition 5.1.5. *For invertible curve singularities, the genus g, the number of boundary components k of the Milnor fiber M_W is given as follows. Also the* *quotient* $[\overline{M}_W/G_W]$ *is an orbifold projective line* $\mathbb{P}^1_{a,b,c}$*:*

$$
\begin{aligned}
&\textit{(Fermat)} \quad g=\tfrac{\mu_F+1-d}{2}, k=d, \quad (a,b,c)=(p,q,\frac{pq}{d}),\ \ d=gcd(p,q)\\
&\textit{(Chain)} \quad g=\tfrac{\mu_C-d}{2}, k=d+1, \quad (a,b,c)=(pq,q,\frac{pq}{d}),\ \ d=gcd(p-1,q)\\
&\textit{(Loop)} \quad g=\tfrac{\mu_L-1-d}{2}, k=d+2, \quad (a,b,c)=(pq-1,pq-1,\frac{pq-1}{d}),\\
&\qquad\qquad d=gcd(p-1,q-1)
\end{aligned}
$$

Here, c vertex for Fermat, a,c-vertices for Chain, a,b,c-vertices for Loop type are punctures.

Proof. It is well-known that the number of boundary components are the same as the number of irreducible factors of W. (Recall that for sufficiently small r and $0<\epsilon \ll r$, the link$W^{-1}(0)\cap S_r^{2m-1}$ and $W^{-1}(\epsilon)\cap S_r^{2m-1}$ are diffeomorphic and note that each factor of W gives a boundary component for $W^{-1}(0)$). For Fermat type, x^p+y^q factors into d factors for $d=gcd(p,q)$. For Chain type, since $x^p+xy^q=x(x^{p-1}+y^q)$, $C_{p,q}$ has $d+1$ factors with $d=gcd(p-1,q)$. For loop type, since $x^py+xy^q=xy(x^{p-1}+y^{q-1})$, $L_{p,q}$ has $d+2$ factors with $d=gcd(p-1,q-1)$.

To compute the genus, note that M_W is obtained by removing k punctures from $\overline{M}_W$. Hence, Euler charactoristic $\mathscr{E}(M_W)=\mathscr{E}(\overline{M}_W)-k$. But M_f has the homotopy type of bouquet of μ-circles for the Milnor number μ, and its Euler charactoristic $\mathscr{E}(M_W)=1-\mu$. Therefore, the genus of $\overline{M}_W$ (and hence M_W) is obtained from $2-2g-k=1-\mu$ or $g=(\mu+1-k)/2$.

Now, to find the quotient orbifold $[\overline{M}_W/G_W]$, we first find there are exactly three orbits (of G_W) with non-trivial stabilizer in $\overline{M}_W$ and show that the quotient has genus zero using orbifold-Euler-characteristic. We will use the fact that $\mathscr{E}(\overline{M}_W)/|G_W|$ equal the orbifold Euler-characteristic of $[\overline{M}_W/G_W]$.

Let us consider the Fermat case. Orbits of $[(0,1)]$ and $[(1,0)]$ gives two singular orbits of $\mathbb{Z}/p\oplus\mathbb{Z}/q$-action on M_F. They have stabilizers $\mathbb{Z}/p,\mathbb{Z}/q$ respectively. For $d=gcd(p,q)$, we have d punctures and $\mathbb{Z}/p\oplus\mathbb{Z}/q$ acts transitively on them. So the quotient has three orbifold points $(a,b,c)=(\mathbb{Z}/p,\mathbb{Z}/q,\mathbb{Z}/(pq/d))$.

To see that the quotient is $\mathbb{P}^1_{a,b,c}$,

$$\mathscr{E}(\overline{M}_W) = 2 - 2g = k - 1 - \mu = d - 1 - (p-1)(q-1)$$

Note that it equals $|G| \cdot \mathscr{E}_{orb}(\mathbb{P}^1_{a,b,c})$ which is

$$(pq) \cdot (\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1) = pq \cdot (\frac{1}{p} + \frac{1}{q} + \frac{d}{pq} - 1)$$

This proves the claim for the Fermat case.

The other cases are similar. For the chain case, the orbit of $(1,0)$ has stabilizer $\mathbb{Z}/p$. The other two orbifold points come from punctures. Note that $C_{p,q}$ is a product of x and $x^{p-1} + y^q$. It is easy to see that G_W action preserves each branches $x = 0$ as well as $x^{p-1} + y^q = 0$. Hence the puncture corresponding to the branch x has the full group G_f as a stabilizer and the other d punctures (for the factors of $x^{p-1} + y^q = 0$ with $d = gcd(p-1, q)$) are acted by G_f in a transitive way. Therefore, the orbifold point has stabilizer $\mathbb{Z}/(pq/d)$. For the loop type, M_W has no fixed points of G_W-action, and the punctures for factors $x, y, x^{p-1} + y^{q-1}$ form three orbits with stabilizer $\mathbb{Z}/(pq-1), \mathbb{Z}/(pq-1), \mathbb{Z}/((pq-1)/d)$. This finishes the proof. □

5.2 Orbifold covering

In the previous section, we observed that G_W acts on the Milnor fiber M_W to produce the following regular orbifold covering

$$\overline{M}_W \to \mathbb{P}^1_{a,b,c}.$$

Given a Riemann surface, there can be two non-equivalent group actions with isomorphic quotient space(see Broughton [Bro91] for example). Hence, to determine the G_f-action explicitly, we find an explicit group homomorphism

$$\phi : \pi_1^{orb}\left(\mathbb{P}^1_{a,b,c}\right) \to G_W \tag{5.2.1}$$

from orbifold fundamental group onto the symmetry group G_W. For the kernel $\Gamma = \mathrm{Ker}(\phi)$, $\overline{M}_W$ is an orbifold covering of $\mathbb{P}^1_{a,b,c}$ corresponding to the kernel Γ with deck transformation group G_W.

We use the following presentation of the orbifold fundamental group of $\mathbb{P}^1_{a,b,c}$

$$\pi_1^{orb}\left(\mathbb{P}^1_{a,b,c}\right) = \left\langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^a = \gamma_2^b = \gamma_3^c = \gamma_1\gamma_1\gamma_3 = 1 \right\rangle \tag{5.2.2}$$

Here, γ_1 is a small loop going around $0 \in \mathbb{P}^1$. γ_2 is for $1 \in \mathbb{P}^1$ and γ_3 is for $\infty \in \mathbb{P}^1$. Later on, this presentation will serve as an additional grading on a Floer theory.

Proposition 5.2.1. *The homomorphism* (5.2.1) *is given as follows.*

1. *(Fermat) For the covering* $M_{F_{p,q}} \to \mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}$, *we have*

$$\phi(\gamma_1) = (1,0),\ \phi(\gamma_2) = (0,1),\ \phi(\gamma_3) = (-1,-1) \in \mathbb{Z}/p \oplus \mathbb{Z}/q$$

2. *(Chain) For the covering* $M_{C_{p,q}} \to \mathbb{P}^1_{pq,q,\frac{pq}{\gcd(p-1,q)}}$, *we have*

$$\phi(\gamma_1) = 1,\ \phi(\gamma_2) = -p,\ \phi(\gamma_3) = p-1 \in \mathbb{Z}/pq$$

3. *(Loop) For the covering* $M_{L_{p,q}} \to \mathbb{P}^1_{pq-1,pq-1,\frac{pq}{gcd(p-1,q-1)}}$, *we have*

$$\phi(\gamma_1) = 1,\ \phi(\gamma_2) = -p,\ \phi(\gamma_3) = p-1 \in \mathbb{Z}/(pq-1)$$

Let us give the proof in each cases separately.

5.2.1 Fermat type $F_{p,q}$

$M_{F_{p,q}}$ is a locus of an equation $x^p + y^q = 1$. We regard them as a Riemann surface of a multivalued function

$$y = (1-x^p)^{\frac{1}{q}}, \quad k = 0, \dots q-1$$

with q branch points $x_i = e^{\frac{2k\pi i}{q}}$ (for $i = 0, \cdots, q-1$). We connect branch point x_i with ∞ by a ray $\{re^{\frac{2k\pi i}{q}} \mid r \geq 1\}$. With this branch cut, $M_{F_{p,q}}$ is a q sheeted covering of a complex plane $\mathbb{C}$.

A fundamental domain of the quotient is the following "pizza" shape domain.

$$\left\{x = re^{\theta} | 0 \leq r,\ 0 \leq \theta \leq \frac{2\pi}{p}\right\} \tag{5.2.3}$$

There are three distinguished paths $\gamma_i : [0,1] \to \mathbb{C}$.

- $\gamma_1(t) = \epsilon \cdot e^{\frac{2\pi i t}{p}}$, $(0 < \epsilon \ll 1)$, a small path around the origin.
- $\gamma_2(t) = 1 + \epsilon \cdot e^{2\pi i t}$, $(0 < \epsilon \ll 1)$ a small circle around the branch points.
- $\gamma_3(t) = Re^{\frac{-2\pi i t}{p}}$, $(R \gg 1)$ a boundary cirle with opposite orientation.

These are orbifold loops that correspond to generators of $\pi_1\left(\mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}\right)$ in 5.2.2.

Let us find the the homomorphism (5.2.1) for the Fermat case. Recall that we realize $F_{p,q}$ as a q sheeted covering of $\mathbb{C}$. Label those sheets by natural numbers from 1 to q so that the crossing branch cuts increases the label number by $+1$. Each sheet has p copies of the fundamental domain. We put the label i_j on the following copies of it;

$$\left\{x = re^{\theta} | 0 \leq r,\ \frac{2(j-1)\pi}{p} \leq \theta \leq \frac{2j\pi}{p}\right\} \subset i\text{th sheet.}$$

In this setup, we can write down the monodromy representation of the fundamental group to the group of permutation of the set of labels $\{i_j | 0 \leq i \leq q,\ \ 0 \leq j \leq p\}$.

$$\begin{aligned}
\phi : \pi_1\left(\mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}\right) &\to S_{pq} \\
\gamma_1 &\mapsto (1_1, 1_2, \ldots, 1_p)(2_1, 2_2, \ldots, 2_p) \cdots (q_1, q_2, \ldots, q_p) \\
\gamma_2 &\mapsto (1_1, 2_1, \ldots, q_1)(1_2, 2_2, \ldots, q_2) \cdots (1_p, 2_p, \ldots, q_p) \\
\gamma_3 &\mapsto (\gamma_1 \circ \gamma_2)^{-1}
\end{aligned}$$

The image of this representation is generated by γ_1 and γ_2, isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/q$. Moreover, it is compatible to the diagoal symmetry group action.

- γ_1 is a rotation of each sheets by $\frac{2\pi}{p}$. It corrsponds to a diagonal action $x \to e^{\frac{2i\pi}{p}} \cdot x, \;\; y \to y.$
- γ_2 is a rotation of each sheets by $\frac{2\pi}{q}$ so it corrsponds to a diagonal action $x \to x, \;\; y \to e^{\frac{2i\pi}{p}} \cdot y$

5.2.2 Chain type $C_{p,q}$

$M_{C_{p,q}}$ is a locus of an equation $x^p + xy^q = 1$. We regard them as a Riemann surface of a multivalued meromorphic function

$$y = \left(\frac{1-x^p}{x}\right)^{\frac{1}{q}}, \qquad k = 0, \dots q-1.$$

This function has a q zero branch points $x = e^{\frac{2k\pi i}{q}}$ and a single pole branch point $x = 0$.

We connect each branch points with ∞ by rays as before. Also, we overlap a ray from a pole $x = 0$ and a ray from a zero $x = 1$. Because they are coming out of different sources, they cancel each others on the overlap. With this choice of branch cuts, $M_{C_{p,q}}$ is a q sheeted covering of $\mathbb{C}^*$.

A fundamental domain of the quotient $M_{C_{p,q}}/G_{C_{p,q}}$ can be taken as the same domain (5.2.3) but in $\mathbb{C}^*$ and orbifold loops $\gamma_1, \gamma_2, \gamma_3$ are the same as in the Fermat cases. Hence γ_1, γ_3 are the loops around the orbifold punctures.

Due to the branch cut along the line segment $[0,1]$ on the real axis, monodromy representation is different from the Fermat cases. It is not hard to see that we get the following symmetric group representation

$$\begin{aligned}
\phi : \pi_1\left(\mathbb{P}^1_{pq,q,\frac{pq}{gcd p-1,q}}\right) &\to S_{pq} \\
\gamma_1 &\mapsto (1_1, 1_2, \dots, 1_p, q_1, q_2, \dots, 3_p, 2_1, 2_2, \dots, 2_p) \\
\gamma_2 &\mapsto (1_1, 2_1, \dots, q_1)(1_2, 2_2, \dots, q_2)\cdots(1_p, 2_p, \dots, q_p) \\
\gamma_3 &\mapsto (\gamma_1 \circ \gamma_2)^{-1}.
\end{aligned}$$

Unlike the Fermat case, $\phi(\gamma_1)$ generates $\phi(\gamma_2)$ by the relation $\phi(\gamma_2) = \phi(\gamma_1)^{-p}$. Therefore the image of ϕ is generated by γ_1, isomorphic to $\mathbb{Z}/pq$. Notice that

$\phi(\gamma_1)$ rotates each sheet by $\frac{2\pi}{p}$ and change the label of a sheet by $+1$ if you apply it by $-p$ times. It corresponds to a diagonal action

$$x \to e^{\frac{2\pi i}{p}} \cdot x, \quad y \to e^{\frac{-2\pi i}{pq}} \cdot y$$

which is a generator of the $\mathbb{Z}/pq$-action.

5.2.3 Loop type $L_{p,q}$

$M_{L_{p,q}}$ is a locus of an equation $x^p y + xy^q = 1$. As we can not realize $L_{p,q}$ as a Riemann surface of a single function, we work with the following parametrization by $z \in \mathbb{C}$

$$x = \left(\frac{z^q}{1-z}\right)^{\frac{1}{pq-1}}, \quad y = \left(\frac{(1-z)^p}{z}\right)^{\frac{1}{pq-1}}, \qquad k = 0, \ldots pq-2$$

with two branch points $z = 0, 1$.

The point $z = 0$ is an order q zero for x and order 1 pole for y. Likewise, the point $z = 1$ is an order 1 pole for x and order p zero for y. We connect these two with ∞ by half lines and let the one from the origin overlaps the one from $z = 1$. Although it is a Riemann sureface of two multivalued functions rather then a single one, we can still $M_{L_{p,q}}$ is now a $pq-1$ sheeted covering of a z-plane $\mathbb{C} \setminus \{0, 1\}$ as follows.

A fundamental domain is the whole z-plane minus two points $z = 0, 1$. The three distinguished paths are

- $\gamma_1(t) = \epsilon \cdot e^{2\pi i t}$, $(0 < \epsilon \ll 1)$, a small circle around $z = 0$.
- $\gamma_2(t) = 1 + \epsilon \cdot e^{2\pi i t}$, $(0 < \epsilon \ll 1)$ a small circle around $z = 1$.
- $\gamma_3(t) = Re^{-2\pi i t}$, $(R \gg 1)$ a boundary circle with opposite orientation.

Let us compute the monodromy representation. Whenever we cross brach cuts inside z-plane, we change a covering sheet for x and y both. Each of them has $pq-1$ possibilities, so there are $(pq-1) \times (pq-1)$ different sheets. Let's label them by (i, j), $i, j = 1, \ldots, pq-1$. But we don't need all of them because

π_1-orbit of $(1,1)$ consists of only $pq-1$ sheets among them. The monodromy representation of π_1^{orb} is written as

$$\begin{aligned}\phi : \pi_1^{orb}\Big(\mathbb{P}^1_{pq-1,pq-1,\frac{pq}{gcd(p-1,q-1)}}\Big) &\to S_{pq-1}\times S_{pq-1}\\ \gamma_1 &\mapsto (+q,-1) : (a,b) \to (a+q, b-1)\\ \gamma_2 &\mapsto (-1,p) : (a,b) \to (a-1, b+p)\\ \gamma_3 &\mapsto (\gamma_1\circ\gamma_2)^{-1}\end{aligned}$$

Since $\phi(\gamma_2) = \phi(\gamma_1)^{-p}$, the image of this representation is generated by γ_1 and isomorphic to $\mathbb{Z}/pq-1$. Notice that γ_1 corresponds to the following element

$$x \to e^{\frac{2q\pi i}{pq-1}}\cdot x, \quad y \to e^{\frac{-2\pi i}{pq-1}}\cdot y$$

of maximal diagonal symmetry group.

5.3 Equivariant tessellation of Milnor fibers

This chapter is a work of [Jeo19]. Using the equator of $\mathbb{P}^1_{a,b,c}$ containing three orbifold points, we can divide the orbi-sphere into two cells. From the orbifold covering $\overline{M}_W \to \mathbb{P}^1_{a,b,c}$, and considering lifts of these two cells, we obtain a tessellation of Milnor fibers of invertible curve singularities. In this section, we give a combinatorial description of the tessellation of $\overline{M}_W$ as well as G_W-action on it.

Consider a $2m$-gon whose boundary edges are labelled by $a_1,\cdots,a_{2m}$ ordered and oriented in a counterclockwise way. We say edges are identified as $\pm(2p-1)$ pattern if a_{2k} and $(a_{2k-(2p-1)})^{op}$ are identified, and a_{2k-1} and $(a_{2k-2p})^{op}$ are identified for any k. (Here indices are modulo $2m$, and a^{op} is the orientation reversal of the edge). Note that even and odd numbered edges play different roles. See Figure 5.3 (B) for 16-gon identified with ± 7 pattern.

Theorem 5.3.1. *Compactified Milnor fiber $\overline{M}_W$ and G_W on it are explicitly described as follows*

1. *(Fermat) $\overline{M}_{F_{p,q}}$ is given by $(2pq-2q)$-gon with edges identified as $\pm(2p-1)$*

pattern. An odd numbered edge corresponds to an oriented path from c-vertex to b-vertex in the quotient.

2. *(Chain) $\overline{M}_{C_{p,q}}$ is given by $(2pq)$-gon with edges identified as $\pm(2p-1)$ pattern. An odd numbered edge corresponds to an oriented path from b-vertex to c-vertex in the quotient.*

3. *(Loop) $\overline{M}_{L_{p,q}}$ is given by $2(pq-1)$-gon with edges identified as $\pm(2p-1)$ pattern. An odd numbered edge corresponds to an oriented path from b-vertex to c-vertex in the quotient.*

Proof. Recall that we have $[\overline{M}_W/G_W] = \mathbb{P}^1_{a,b,c}$ from Proposition 5.1.5. Let H be the universal cover of $\overline{M}_f$ or equivalently that of $\mathbb{P}^1_{a,b,c}$. We have $\pi_1^{orb}(\mathbb{P}_{a,b,c})$ action on H. Let F be a fundamental domain in H for this action as in the Figure 5.1 where the angle is measured in S^2 or $\mathbb{R}^2$ or $\mathbb{H}$ depending on the universal cover. Here x_1, x_2, x_3 project down to a, b, c orbifold points and at x_1, x_2 we have the full cone angle but the cone angle for x_3 is divided into half for x_3 and x_3'. Also, we will use Proposition 5.2.1 which describes the relation between generators $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1^{orb}(\mathbb{P}_{a,b,c})$ and G_f.

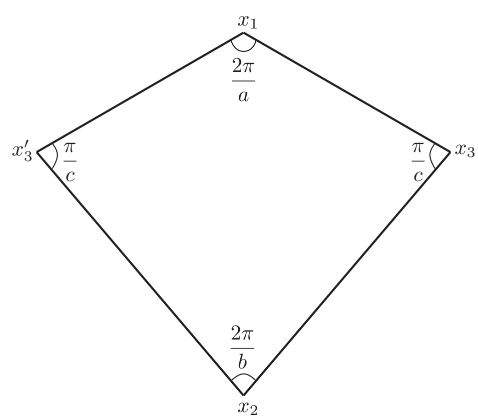


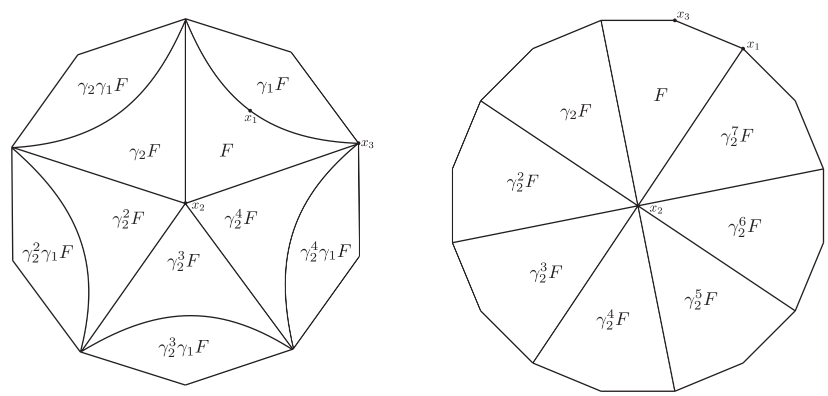
Figure 5.1: Fundamental domain of $\mathbb{P}^1_{a,b,c}$ in $\mathbb{H}$

For the Fermat case, consider $\gamma_1, \gamma_2 \in \pi_1^{orb}(\mathbb{P}_{a,b,c})$ and collect the following $p \times q$ copies of F to define a polygon

$$P := \left\{ \gamma_2^i \gamma_1^j F \mid 0 \le i \le p-1, 0 \le j \le q-1 \right\}.$$

First, P is a fundamental domain of $\overline{M}_W$ since $G_W = \mathbb{Z}/p \times \mathbb{Z}/q$ and $\phi(\gamma_1) = (1,0), \phi(\gamma_2) = (0,1)$ are the generators of G_W by Proposition 5.2.1.

Also, one can check that P is a $(2pq-2q)$-gon in the following way. First, $\{\gamma_1^j F\}$ for $j = 0, \cdots, p-1$ can be glued counter-clockwise way around the vertex x_1 of Figure 5.1 to form a $2p$-gon, say Q. Then, by applying $\{\gamma_2^i\}$ for $i = 0, 1, \cdots, q-1$ to Q, we get q-copies of Q glued around the vertex x_2 to form a $(2pq-2q)$-gon and this is exactly P. Because $2q$ edges meeting the vertex x_2 become interior edges, number of boundary edges decrease by $2q$ from $2pq$. See Figure 5.2 (A) for the case of $\overline{M}_{F_{2,5}}$, where Q is given by the union of F and $\gamma_1 F$ and P is the 10-gon.



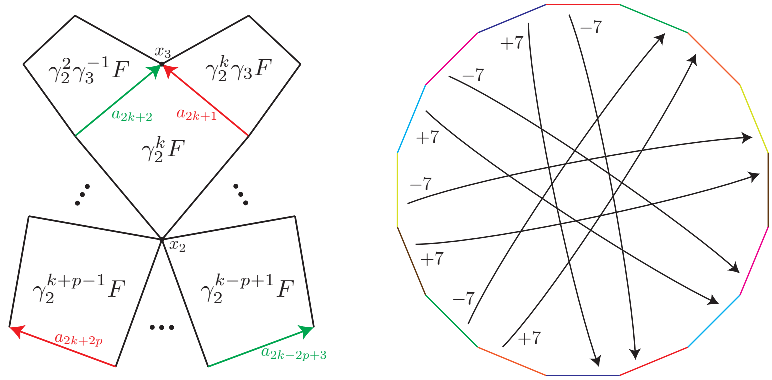
(a) $\overline{M}_{F_{2,5}}$ case

(b) $\overline{M}_{C_{4,2}}$ case

Figure 5.2: Tessellations of A_4 and D_5 singularities

Note that P has an obvious induced $\mathbb{Z}/p \times \mathbb{Z}/q$-action. Namely, $\mathbb{Z}/p$-action is the $\frac{2\pi}{p}$ rotation around the center of P, and $\mathbb{Z}/q$-action is the $\frac{2\pi}{q}$ rotation of every copy of Q around their centers. In fact, for the edge e which is given by the intersection $(\gamma_2^i Q) \cap (\gamma_2^{i+1} Q)$, action by $\mathbb{Z}/q$ on e will depend on whether we interpret e as an element of $(\gamma_2^i Q)$ or $(\gamma_2^{i+1} Q)$. But the boundary identification of P is exactly the relations that make two $\mathbb{Z}/q$-actions on e to agree with each other. One can check that it gives $\pm(2p-1)$ identification on ∂P. This proves the proposition for the Fermat case.

Next, let us discuss the chain type. Recall that we have $G_{C_{p,q}} = \mathbb{Z}/pq$, and $\phi(\gamma_1) = 1 \in \mathbb{Z}/pq$ is the generator. We take the following pq-copies of F to de-



(a) Boundary identification (b) ±7 pattern

Figure 5.3: Chain cases

fine a $2pq$-gon:

$$P := \left\{\gamma_1^i F \mid 0 \leq i \leq pq-1\right\},$$

which is a fundamental domain for the Milnor fiber. To find the boundary identification, we consider additional tiles next to P. To see how two boundary edges from $\gamma_1^k F$ are identified to the remaining edges, we consider the addition rotation action around vertex for x_2. Namely, consider $\gamma_2^{-1}\gamma_1^k F$ and $\gamma_2\gamma_1^k F$. Since $\phi(\gamma_2) = -p$, we have $\phi(\gamma_2^{-1}\gamma_1^k) = \phi(\gamma^{p+k})$. Therefore, $\gamma_2^{-1}\gamma_1^k F$ can be identified with $\gamma^{p+k}F$ as a tile in the Milnor fiber. From this, we can deduce that x_2x_3 edge of $\gamma_1^k F$ should be identified with x_2x_3' edge of $\gamma_1^{k+p}F$. See Figure 5.3 (A). In terms of edges of P, this is $+(2p-1)$ identification. From the same argument for $\gamma_2\gamma_1^k F$, we find that x_2x_3' edge of $\gamma_1^k F$ should be identified with $-(2p-1)$ pattern. This proves the chain cases.

For the loop type, we can proceed similarly as in the chain case. We take

$$P := \{\gamma_3^i F \,|\, 0 \leq i \leq pq-2\}$$

and we get the same identification as in the chain case from Proposition 5.2.1.

□

Remark 5.3.2. *We remark that* $\overline{M}_W$ *is a sphere for* $F_{2,2}$*, and is a torus for* $F_{3,2}, F_{4,2}$*,* $C_{2,2}, C_{3,2}, L_{2,2}$ *and a higher genus surface for the rest of the cases. For the last case, universal cover can be taken as the hyperbolic plane* $\mathbb{H}$ *and there exists a*

Fuchsian group Γ *such that* $\mathbb{H}/\Gamma \simeq \overline{M}_W$. *Furthermore, a finite group* G *acts on* M *if and only if there exist a Fuchsian group* Γ' *and a surjective homomorphism* $\phi : \Gamma' \to G_W$ *with torsion-free kernel* Γ *such that* $M \simeq \mathbb{H}/\Gamma$ *and* $M/G \simeq \mathbb{H}/\Gamma'$.

Chapter 6

Equivariant Floer theory of a Milnor fiber

In this section, we apply our general Floer theories to an orbifold quotient $[M_W/G_W]$ of an invertible curve singularities.

6.1 Hamiltonian

$[M_W/G_W]$ is a three-punctured sphere with orbifold/punctures at $\{0,1,\infty\} \in \mathbb{P}^1$. At each point, we are using one of the following orbifold/puncture chart with G-equivarant ;

- a disc chart
$$\left(D^2, \mathbb{Z}/n, dx \wedge dy\right)$$
with coordrdinate $w = x + iy$. $\mathbb{Z}/n$ acts on D by rotation.

- a puncture chart
$$\left(S^1 \times [0,\infty), \mathbb{Z}/n, dr \wedge d\theta\right)$$
with coordinate system $w = e^{r+\theta i}$. $\mathbb{Z}/n$ acts on $S^1 \times [0,\infty)$ by rotation on θ coordinate.

To keep an orbifold information of disc charts, we further restrict our positive Hamiltonian H to be G_W-equivariant and of the following form;

- outside puncture charts, H is Morse and C^2-small enough so that it does not have a time-1 periodic orbit outside of a cylinder chart.
- at a puncture chart, H is quadratic w.r.t r:

$$H = ar^2 + b, \quad (a > 0).$$

- at a disc chart, H is a function of r and have the unique Morse minimum at $0 \in D$. For example,

$$H = \epsilon - \frac{\epsilon}{2} e^{-r^2}, \quad (0 < \epsilon << 1)$$

This class of hamiltonian is $\mathbb{Z}/n$- equivariant and its pull-back is still quadratic at the end. We sometimes using a "uniformization"

$$x = w^n$$

to describe a complex neighborhood of an orbifold point. Be aware, under this coordinate transform, a pull-back of quadratic Hamiltonians is no longer quadratic. Whenever we use such notation we implicitly replace the neighborhood of the origin to above disc/cylinder charts.

6.2 Ω- and H^1-grading

We describe two grading systems on the Floer theory on $[M_W/G_W]$. It is a slight generalization of [Sei11].

An assumption $c_1(M) = 0$ was crucial for $\mathbb{Z}$ gradings on symplectic cohomology or wrapped Floer cohomology. Unfortunately, an orbifold canonical bundle $K_{[M_W/G]}$ is never trivial. Therefore it is reasonable to expect that the Floer theory on $[M_W/G]$ cannot have a compatible $\mathbb{Z}$-grading. As a partial remedy, we use a holomorphic volume form with specific poles. Up to constant, there is a unique holomorphic volume form Ω on $\mathbb{P}^1$ with poles of order one at $0, 1 \in \mathbb{P}^1$.

This choice provides a trivialization of a tangent bundle $T_{[M_W/G]}$ away from $0, 1$ and ∞. Because of our choice of Hamiltonians, its time-1 orbits are always disjoint from $0,1,\infty \in \mathbb{P}^1$ after adding a small time-dependent perturbation term. Therefore, each hamiltonian orbits still carries an honest cohomological conley-Zehnder index. we use this integer as a degree.

We can put a grading on a Lagrangians and Hamiltonian chords between them in a similar fashion. For a Lagrangian L which is oriented and away from $0,1,\infty \in \mathbb{P}^1$, we get a phase map w.r.t Ω;

$$\overline{\phi}_L : L \to S^1 \tag{6.2.1}$$

$$\overline{\phi}_L(x) = \frac{\Omega(X)}{|\Omega(X)|} \tag{6.2.2}$$

where X is a nonvanishing vector field on TL pointing positive direction.

Definition 6.2.1. *An Ω-grading on L is a choice of lift*

$$\phi_L : L \to \mathbb{R}$$

of a phase map $\overline{\phi}$. An Ω-graded Lagrangian L is a Lagrangian submanifold with a specific choice of Ω- grading ϕ_L.

For any time-1 hamiltonian chords $a \in \chi(L_0, L_1)$ between graded Lagrangian submanifold, there is a unique homotopy class of Lagrangian path from $T_{L_0,a(0)}$ to $T_{L_1,a(1)}$ compatible to the gradings. The absolute Maslov index $\mu_M(a)$ is now well defined, and we use it as a degree of a.

A discrepancy occurs when we consider a moduli space of discs. A standard index formula starts to read an intersection number of a holomorphic maps and pole divisor of Ω. Let

$$\overline{\mathcal{M}}_{m;n,1;[u]}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n,a_0)$$

be a sub-moduli space of $\overline{\mathcal{M}}_{m;n,1;u}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n,a_0)$ whose relative class is $[u]$. Then a standard index formula is now read

$$\begin{aligned}\dim_{\mathbb{R}}\overline{\mathcal{M}}_{m;n,1;[u]}(\gamma_1,\dots,\gamma_m;a_1,\dots,a_n,a_0) = (2m+n-2)\\ +\deg a_0 - \sum_{i=1}^{n}\deg a_i - \sum_{j=1}^{n}\deg\gamma_j\\ +2\big(\deg(u,0)+\deg(u,1)\big)\end{aligned}$$

The dimension of our moduli space may differ in even numbers. It breaks a $\mathbb{Z}$-grading into $\mathbb{Z}/2$-grading.

Meanwhile, there is a topological grading coming from an orbifold cohomology. Recall we use a notation $\gamma_0,\gamma_1,\gamma_\infty$ to denote a homotopy class of loops winding orbifold point $0,1$ or ∞ respectively. We use the same notation for corresponding homology class. We get

$$H^1_{orb}([M_W/G_W]) \simeq \begin{cases} \mathbb{Z}\langle\gamma_1,\gamma_2,\gamma_3\rangle/\{p\gamma_1=q\gamma_2=\gamma_1+\gamma_2+\gamma_3=0\} & (W=x^p+y^q)\\ \mathbb{Z}\langle\gamma_1,\gamma_2,\gamma_3\rangle/\{q\gamma_1=\gamma_1+\gamma_2+\gamma_3=0\} & (W=x^p+xy^q)\\ \mathbb{Z}\langle\gamma_1,\gamma_2,\gamma_3\rangle/\{\gamma_1+\gamma_2+\gamma_3=0\} & (W=x^py+xy^q)\end{cases}$$

Notice that any symplectic cochains, including Morse critical point, can be considered as an element of H^1_{orb}. Moreover, if the Lagrangian submanifold L is simply connected, elements of $CW^\bullet(L,L)$ can also be labeled by H^1_{orb}. Let's call it an H^1*-grading*. The Floer theoretic operation uses pseudo-holomorphic curves whose homological boundary is a difference of homology class of an output and inputs. Therefore, we have

Lemma 6.2.2. *A pseudo-holomorphic curve operation is homogeneous with respect to an H^1-grading.*

6.3 Orbifold wrapped Fukaya category

For a definition of equivariant Floer cohomology, we follow [Sei11] closely.

The collection of Lagrangians $\mathscr{W}$ we consider are Lagrangians $L\in[M_W/G_W]$ such that

- it is conical at the end;

- it carries a grading and spin structure (=an additional real line bundle);
- it is away from a singular locus $\{0,1,\infty\}$.
- its G_W-orbit $\{g\cdot\tilde{L} : g\in G_W\}$ of a lift $\tilde{L}$ intersect transversally to each other only finitely many times;

For two such Lagrangians $L_0, L_1 \in \mathscr{W}$, *a G_W-equivariant Floer cochain complex* is defined by

$$CW^{G_W,\bullet}(L_0,L_1) := \left(\bigoplus_{g,h\in G_W} CW^{\bullet}\left(g\cdot\widetilde{L_0}, h\cdot\widetilde{L_1}\right)\right)^{G_W}$$

where $\widetilde{L_i}$ is a lift of L_i. If there is no confusion, we omit a group notation G_W and simply write $CW^{\bullet}(L_0,L_1)$. In a similar fashion,

Definition 6.3.1. *An orbifold wrapped Fukaya category*

$$\mathscr{W}\mathscr{F}([M_W/G_W])$$

consists of;

1. *a set of objects $\mathscr{W}$;*
2. *space of morphisms are $CW^{G_W,\bullet}(L_0,L_1)$, graded by the parity of* $\deg a$*;*
3. *an A_∞ structure map is a G-invariant part of the m_k-operation of $\mathscr{W}\mathscr{F}(M_W)$.*

Remark 6.3.2. *As noticed in [Sei11], an explicit perturbation scheme extends to the equivariant case without any serious problem. Such perturbation data are inhomogeneous terms of the pseudo-holomorphic curve equation which vary on the **domain** of the curve instead of the target. The group G acts only on the target space M. Therefore we have enough freedom to extend perturbation data in an equivariant way.*

Equivariant wrapped Floer cohomology is indeed a Floer cohomology of an orbifold. We have a direct sum decomposition

$$\bigoplus_{h\in G_W} CW^\bullet(L_0,L_1)^h \simeq CW^\bullet(L_0,L_1) \tag{6.3.1}$$

$$\gamma \mapsto \sum_{g\in G_W} g(\gamma), \tag{6.3.2}$$

$$CW^\bullet(L_0,L_1)^h := CW^\bullet\left(\widetilde{L_0},\ h\cdot\widetilde{L_1}\right). \tag{6.3.3}$$

Therefore, each chord carries an information of an 'arrow' of an orbifold $[M_W/G_W]$. m_k operation respects this additional index, which means it restricts to

$$m_k : \bigotimes_{g_i\in G} CW^\bullet(L_{i-1},L_i)^{g_i} \to CW^\bullet(L_0,L_k)^{\prod_i g_i}$$

Conversely, we have an action of a character group $\hat{G}_W = \mathrm{Hom}(G_W,\mathbb{C}^*)$ of G on the LHS given by

$$\phi(\gamma) := \phi(h)\cdot\gamma \ \text{ when } \ \gamma \in CW^\bullet\left(\widetilde{L_0},\ h\cdot\widetilde{L_1}\right).$$

It is clear from the definition that A_∞ operation is equivariant with respect to this action, and we obtain

$$\left(\bigoplus_{g,h\in G_W} CW^\bullet\left(g\cdot\widetilde{L_0}, h\cdot\widetilde{L_1}\right)\right) \simeq CW^\bullet(L_0,L_1)\ltimes \hat{G}.$$

Notice that $\hat{G}_W = G_{W^T}$, a Berglund-Hübsch dual group of G_W.

A holomorphic disc $u : S \to M_W$ defining A_∞ structure can be considered a **smooth** holomorphic orbi-discs

$$\overline{u} : S \to [M_W/G_W]$$

ramified at orbifold points accordingly. Conversely, because S has a vanishing orbifold fundamental group, all smooth holomorphic orbi-discs lifts to M_W in a G_W-equivariant manner. Therefore,

Fukaya category of $[M_W/G_W] \leftrightarrow G_W$-equivariant Fukaya category of M_W.

6.4 Orbifold symplectic cohomology

We define an orbifold version of symplectic cohomology for this particular case.

Recall our autonomous Hamiltonian H has orbifold points as its Morse minimum. We also restrict a class of time-dependent perturbation. Whenever we add C^2 small, S^1-dependent function F to autonomous Hamiltonian H, we assume $F = 0$ in a sufficiently small disc chart so that orbifold points is still a Morse critical point of $H + F$.

We define *an orbifold symplectic cochain complex* as

$$CH^\bullet([M_W/G_W]) := \bigoplus_{\gamma \in \mathcal{O}} \mathbb{C} \cdot \gamma$$

where

$$\mathcal{O} := \mathcal{O}(H + F)$$

is a time-1 orbits of hamiltonian function H perturbed by F. By definition, a space of orbifold loops is again an orbifold

$$\mathcal{L}([M_W/G_W]) = \left\{(g,\gamma) \mid g \cdot \gamma(0) = \gamma(1)\right\}.$$

where $h \in G$ acts by

$$h \cdot (g,\gamma) = (hgh^{-1}, h\gamma(t)).$$

By our choice of Hamiltonians, an element of $\mathcal{O}$ falls into one of three types;

- **Morse critical point** of $H + F$ without isotropy group.
- **twisted sectors** represented by a constant loop at the origin of each disc chart
$$(\xi, 0) \in \mathcal{L}([D/(\mathbb{Z}/n)]), \quad \xi \in \mathbb{Z}/n;$$
- **Hamiltonian chords** at the end. They are locally represented by a small perturbation of
$$\gamma_k(t) = \left(e^{\frac{2\pi k i t}{n}}, k\right) \subset (S^1 \times [1,\infty), \mathbb{Z}/n).$$

It is $\mathbb{Z}/2$-graded by the parity of a $\deg\gamma$. A degree of Morse critical point and

Hamiltonian chords is its cohomological Conley-Zehnder index as usual. We put a degree of twisted sectors as zero.

Remark 6.4.1. *In the Chen-Ruan cohomology theory, a degree of twisted sector* $(\xi,0)$ *is shifted by a rational number* $\frac{2|\xi|}{n}$*. This is because the action of* ξ *on the orbifold tangent bundle is not trivial. In our case, a holomorphic volume form* Ω *we choose has a pole of order one at the orbifold point, so the action of* ξ *is trivial on it. Therefore a cohomological Conley-Zehnder index of* $(\xi,0)$ *is zero. An effect of an isotropy group is absorbed by* Ω*.*

Its differential d_{CH} is defined as

$$\mathrm{d}_{CH}(\gamma_1) = (-1)^{\deg\gamma}\mathbf{F}_{1,1;0}(\gamma_1).$$

We also define a pair-of-pants product by

$$\gamma_1\cdot\gamma_2 := (-1)^{\deg\gamma_1}\mathbf{F}_{2,1;0}(\gamma_1,\gamma_2)$$

We should point out that a family of smooth pseudo-holomorphic curves may contains an orbifold nodal point. (We avoided this issue in the definition of Fukaya category). Fortunately, we can rule out those contribution in this case.

Lemma 6.4.2. $d_{CH}^2 = 0$*. The product structure induces a ring structure on* d_{CH}*-cohomology.*

Proof. It is enough to show that orbifold nodal degeneration does not affect a standard analysis of codimension 1 boundary strata of moduli spaces. A local model of such degeneration is given by a family

$$z_1z_2 : \left[\mathbb{C}^2/(\mathbb{Z}/n)\right] \to \mathbb{C}$$

Here, a group $\mathbb{Z}/n$ acts on $\mathbb{C}^2$ by $(z_1,z_2)\to(\xi\cdot z_1,\xi^{-1}\cdot z_2)$ with ξ an n-th root of unity. Generic fibers are a smooth cylinder while the zero fiber is an orbifold nodal curve. From this local model, we conclude that orbifold nodal degeneration happens inside a codimension 2 boundary strata of a moduli space of domains. It does not appear in a codimension 1 boundary strata of $\overline{\mathcal{M}}_{1,1;0}(\gamma_1,\gamma_0)$ and $\overline{\mathcal{M}}_{2,1;0}(\gamma_1,\gamma_0)$ if we choose a generic almost complex structure. □

We call a cohomology group an orbifold symplectic cohomology, denoted by

$$SH^\bullet([M_W/G_W]) = H^\bullet\left(CH^\bullet([M_W/G_W]), \mathrm{d}_{CH}\right).$$

It is clear that $CH^\bullet([M_W/G_W])$ again encode an orbifold information. We get a direct sum decomposition

$$CW^\bullet([M_W/G_W]) = \bigoplus_{h\in G_W} CW^\bullet([M_W/G_W])^h,$$

where $CW^\bullet([M_W/G_W])^h$ consists of $(h,\gamma) \in \mathscr{L}([M_W/G_W])$. Differential and product respects this decomposition. A lift of a smooth pseudo-holomorphic cylinder $u \in \mathscr{M}_{1,1;0}(\gamma_+,\gamma_-)$ is a strip

$$\widetilde{u} : Z \to M_W, \tag{6.4.1}$$

$$u(\pm\infty, t) = \gamma_\pm(t), \tag{6.4.2}$$

$$g\cdot u(s,0) = u(s,1), \quad \forall s \in (-\infty,\infty). \tag{6.4.3}$$

rather than a cylinder. Therefore $\gamma_\pm$ must lie in a same direct summand. Similar result hold for a product structure. Namely, it restricts to

$$CH^\bullet([M_W/G_W])^{h_1} \otimes CH^\bullet([M_W/G_W])^{h_2} \to CH^\bullet([M_W/G_W])^{h_1h_2}.$$

In particular, the product structure commutative only because G_W is abelian.

We define a closed-open map

$$\mathrm{CO} : CH^\bullet(M_W/G_W) \to CC^\bullet\left(\mathscr{W}\mathscr{F}([M_W/G_W]), \mathscr{W}\mathscr{F}([M_W/G_W])\right)$$

popsicle operation

$$m^\Gamma_{n,F,\phi}$$

and a new category

$$\mathscr{C}_\Gamma$$

associated to $\Gamma \in SH^\bullet([M_W/G_W])$ in a same manner as before. By the same reason as in 6.4.2, CO is a chain map. Also, $\mathscr{C}_\Gamma$ is an A_∞ category.

6.5 Floer algebra of Seidel's immersed Lagrangian $\mathbb{L}$ and its deformation

Since $[M_W/G_W]$ is an orbifold sphere with three special points, we can consider an immersed circle, called Seidel Lagrangian $\mathbb{L}$ and its A_∞-algebra following Seidel. (See Figure 6.1 and [Sei11]) We breifly recall the algebra structure of $CF^\bullet(\mathbb{L},\mathbb{L})$. It has immersed generators X, Y, Z of odd degree, $\bar{X} = Y \wedge Z, \bar{Y} = Z \wedge X, \bar{Z} = X \wedge Y$ of even degree.

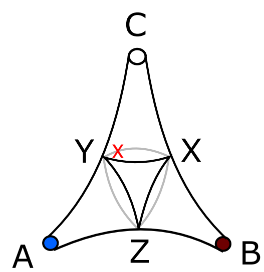


Figure 6.1: Orbifold sphere $\mathbb{P}^1_{a,b,c}$ in the Fermat case with one puncture C

Its Ω and H^1-grading is given by the following table.

	$1_{\mathbb{L}}$	X	Y	Z	$\bar{X}$	$\bar{Y}$	$\bar{Z}$	$[pt] = X \wedge Y \wedge Z$
Ω-grading	0	1	1	-1	2	0	0	1
H^1-grading	0	$-\gamma_2$	$-\gamma_1$	$-\gamma_3$	γ_2	γ_1	γ_3	0

Using the reflection symmetry (take $\mathbb{L}$ to be invariant under the reflection), we can follow [CHL17] to prove that $\mathbb{L}$ is weakly unobstructed and comput its potential function, denoted as $\widetilde{W}$.

Lemma 6.5.1. *Equip $\mathbb{L}$ with a nontrivial spin structure (marked as red crossing in Figure 6.1). Then*

1. *$\mathbb{L}$ is weakly unobstructued.*
2. *$b = xX + yY + zZ$ is a weak bounding cochains with potential $\widetilde{W}$ where*

$$\widetilde{W} = \begin{cases} x^p + y^q + xyz, & \textit{for } F_{p,q} \\ x^q + xyz, & \textit{for } C_{p,q} \\ xyz, & \textit{for } L_{p,q} \end{cases}$$

Remark 6.5.2. *Note that $\widetilde{W}$ is independent of p for the chain, p, q for the loop case. This is not a contradiction because the quotient space $[M_W/G_W]$ is also independent of those indices as well*

Remark 6.5.3. *Since Milnor fibers are exact and $\mathbb{L}$ is exact Lagrangian (since it is homologically trivial), there exist a change of coordinate such that W does not have any area T-coefficient. Therefore, we will omit them in the paper.*

$\mathbb{L}$ is oriented so that edges of the front xyz triangle are oriented counter-clockwise.

Proof. Weakly unobstructedness can be proved exactly the same way as Theorem 7.5 of [CHL17]. To compute $\widetilde{W}$, we fix a generic point and count all polygons whose corners are given by X, Y, Z's. Because of punctures, there are finitely many polygons contributing to $\widetilde{W}$. Also, we are only counting smooth discs, which have lifts to the Milnor fiber. So we can count them in the cover.

In the Fermat case, recall that the Milnor fiber can be obtained by first taking $2p$-gon and taking $2q$-copies of these $2p$-gon's by rotation around the $\mathbb{Z}/2q$ fixed point. Then, we have one $2p$-gon and one $2q$-gon and XYZ-triangle passing through a generic point. See Figure.. Therefore, we have

$$\widetilde{W} = x^p + y^q + xyz$$

In the Chain case, its Milnor fiber is given by a $2pq$-gon with A-puncture at the center and B-vertex and C-puncture as vertices of $2pq$-gon (with $\mathbb{Z}/pq$-action around A). To see the discs, it is more convenient to cut this into pq-pieces along rays connecting A and C. We can glue q of them around the B-vertex ($\mathbb{Z}/q$-fixed point) to obtain $2q$-gon, and as there are p many B-vertices, we get p many $2q$-gons. (see Figure ...) Each $2q$-gon has all the vertices as punctures, and one can check that a rigid holomorphic polygon with boundary on $\mathbb{L}$ has to be contained in one of the $2q$-gon. By inspection we obtain

$$\widetilde{W} = y^q + xyz$$

In the Loop case, all vertices are punctures and one can easily check that the only nontrivial disc is XYZ-triangle. Hence we have

$$\widetilde{W} = xyz$$

□

For weakly unobstructed $\mathbb{L}$, localized mirror functor formalism ([CHL17]) provides a canonical A_∞-functor from Fukaya category of $[M_W/G_W]$ to the matrix factorization category of $\widetilde{W}$.

We recall its definition from [CHL17] to set the notations.

6.6 Localized mirror functor to Matrix factorization category

Let R be a polynomial algebra over the algebraically closed field k of characteristic 0.

Definition 6.6.1. *For $f \in R$ a matrix factorization of f is defined by a pair (P, d) where P is $\mathbb{Z}/2\mathbb{Z}$-graded free R-module and d is an odd degree endomorphism such that $d^2 = f \cdot id$. The dg-category of matrix factorizations $MF_{dg}(f)$ of f is defined as follows. An object of $MF_{dg}(f)$ is a matrix factorization, and* $\hom_{MF_{dg}(f)}(P, P')$ *is given by $\mathbb{Z}/2\mathbb{Z}$-graded R-module maps between P and P', with usual composition $\circ$. A differential d on homogeneous morphisms are defined by*

$$d(\phi) = d_{P'} \circ \phi - (-1)^{deg(\phi)} \phi \circ d_P.$$

It is more convenient to use A_∞-category $\mathcal{MF}(f)$ for mirror symmetry.

Definition 6.6.2. *An A_∞-category $\mathcal{MF}(f)$ is defined as follows. An object of $\mathcal{MF}(f)$ is a matrix factorization of f. For two matrix factorizations P and P', its morphism space is*

$$\hom_{\mathcal{MF}(f)}(P, P') = \hom_{MF_{dg}(f)}(P', P)$$

with A_∞-operations

$$m_1 := d, \ \ m_2(\phi, \psi) := (-1)^{deg(\phi)} \phi \circ \psi, \ \ m_{\geq 3} = 0.$$

Definition 6.6.3. *Let $W^{\mathbb{L}}$ be the disk potential of $\mathbb{L}$. The localized mirror functor $\mathscr{F}^{\mathbb{L}} : Fuk(X) \to MF_{A_\infty}(W^{\mathbb{L}})$ is defined as follows.*

- *For given Lagrangian L, $\mathscr{F}^{\mathbb{L}}(L) := (CF(L,\mathbb{L}), -m_1^{0,b}) =: M_L$.*
- *Higher component*

$$\mathscr{F}_k^{\mathbb{L}} : CF(L_1, L_2) \otimes \cdots \otimes CF(L_k, L_{k+1}) \to MF_{A_\infty}(M_{L_1}, M_{L_{k+1}})$$

  is given by

$$\mathscr{F}_k^{\mathbb{L}}(a_1, \cdots, a_k) := \sum_{i=0}^{\infty} m_{k+1+i}(a_1, \cdots, a_k, \bullet, \overbrace{b, \cdots, b}^{i}).$$

  Here the input $\bullet$ is an element in $M_{L_{k+1}} = CF(L_{k+1}, \mathbb{L})$.

This construction based on Lagrangian Floer theory gives an A_∞-functor.

Theorem 6.6.4 (Theorem 2.19 [CHL17]). *$\mathscr{F}^{\mathbb{L}}$ is a covariant A_∞-functor.*

Chapter 7

Homological mirror symmetry for Milnor fibers of invertible curve singularity

In this section, we consider homological mirror symmetry for Milnor fiber as a symplectic manifold. We will find that G_W-equivariant mirror of M_W is a Landau-Ginzurg model $\widetilde{W}$. By applying Theorem 6.6.4 to the wrapped Fukaya category of $[M_W/G_W]$, we obtain an A_∞-functor, which is shown to give derived equivalence.

Theorem 7.0.1. *We have an A_∞-functor $\mathscr{F}^{\mathbb{L}}$*

$$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_W/G_W]) \to \mathscr{M}\mathscr{F}(\widetilde{W})$$

where $\widetilde{W}$ for Fermat $F_{p,q}$, Chain $C_{p,q}$ and loop $L_{p,q}$ cases are given as

$$\widetilde{W} = x^p + y^q + xyz, \quad x^q + xyz, \quad xyz$$

This functor is fully faithful and gives a derived equivalence between two categories.

Remark 7.0.2. *$\widetilde{W}$ is related to the transposed potential W^T as follows. If we set*

$$g(x, y, z) = z, z - y^p, z - x^{p-1} - y^{q-1}$$

then we have

$$\widetilde{W} = W^T(x, y) + xyg.$$

As we will explain later, if we add monodromy information and take our newly defined A_∞-category, the mirror will be obtained by setting $g = 0$, hence we obtain the matrix factorization of $W^T(x, y)$.

We prove the above theorem in the rest of the section. Although we treat each cases separately, the underlying strategies are basically the same.

7.1 Fermat cases

Recall that $G_W = \mathbb{Z}/p \times \mathbb{Z}/q$ is the maximal diagonal symmetry of $W = x^p + y^q$ and the quotient space $[M_{F_{p,q}}/G_W]$ has a single puncture say C of orbifold order $\frac{pq}{gcd(p,q)}$. Then, for a preimage $\widetilde{C}$ in $M_{F_{p,q}}$, we connect $\widetilde{C}$ and $(1,0)\cdot\widetilde{C}$ by a shortest path $\widetilde{L}_1$ as in the Figure ..., which we take as a non-compact Lagrangian. We denote by L the embedded Lagrangian in $[M_{F_{p,q}}/G_W]$ given by its projection. Denote by $\widetilde{L}$ the set of lifts of L in $M_{F_{p,q}}$, which is exactly $G_W \cdot \widetilde{L}_1$.

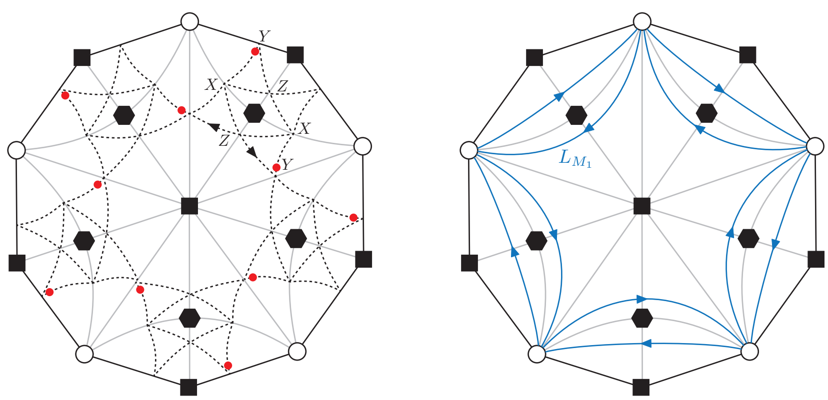


Figure 7.1: Milnor fiber of $F_{4,2}$ and a choice of Lagrangian L and its lifts $\widetilde{L}$

To prove the theorem in Fermat case, we show that G_W copies of L split-generates $\mathscr{W}\mathscr{F}(M_{F_{p,q}})$. Also, we compute the mirror matrix factorization $\mathscr{F}^{\mathbb{L}}(L)$, and show that the functor is fully faithful. Finally, we show that $\mathscr{M}\mathscr{F}(\widetilde{W})$ is split generated by $\mathscr{F}^{\mathbb{L}}(L)$, and this proves the theorem 7.0.1.

Wrapped Fukaya category of a surface is rather well-known Since G_W acts freely on objects, it is easy to see that wrapped Fukaya category of $M_{F_{p,q}}$ has a strict G_W-action, and we have

$$CW^\bullet(L,L) = CW^{G_W,\bullet}(\widetilde{L},\widetilde{L})$$

Lemma 7.1.1. *Wrapped Floer complex $CW^\bullet(L,L)$ is quasi-isomorphic to the following model.*

1. *As a vector space,*
$$CW^\bullet(L,L) \simeq T(a,b)/\mathscr{R}_{F_{p,q}}$$
Here, $T(a,b)$ is a tensor algebra generates by two alphabets a,b. The ideal $\mathscr{R}_{F_{p,q}}$ is defined as
$$\mathscr{R}_{F_{p,q}} =< a\otimes a = \delta_{2,p}, b\otimes b = \delta_{2,q} >$$
$\mathbb{Z}/2$-grading of a,b is odd and this induces $\mathbb{Z}/2$-grading on $T(a,b)/\mathscr{R}_{F_{p,q}}$.

2. *m_1 vanishes and m_2 coincides with the tensor product.*

3. *$m_k(a,\dots,a)$ is zero for $1 \le k < p$ and it is equal to 1 for $k = p$. Likewise, $m_k(b,\dots,b)$ is zero for $1 \le k < q$ and equal to 1 for $k = q$*

Its Ω and H^1-grading is given by the following table.

	1_L	a	b
Ω-*grading*	0	1	1
H^1-*grading*	0	$-\gamma_3$	$-\gamma_3$

Proof. We can choose $\widetilde{L}_1$ so that G_W-orbits of $\widetilde{L}_1$ are disjoint. Therefore $CW^\bullet(L,L)$ consists only of hamiltonian chords at an infinity. Among such chords we choose the following two generators.

- a, the shortest chord $\in CW^\bullet(\widetilde{L}_1,(1,0)\cdot\widetilde{L}_1)$, $\quad (1,0) \in \mathbb{Z}/p\times\mathbb{Z}/q$
- b, the shortest chord $\in CW^\bullet(\widetilde{L}_1,(0,1)\cdot\widetilde{L}_1)$, $\quad (0,1) \in \mathbb{Z}/p\times\mathbb{Z}/q$.

For example, take a rotation of $\widetilde{L}_1$ around the $\mathbb{Z}/p$ fixed point and there is a unique wrapped generator a between these two branches. By abuse of notation, we also denote by a, b the generators in $CW^{G_W,\bullet}(\widetilde{L},\widetilde{L})$ given by the sum of G_W-copies of the above generators.

We can also concatenate them to create new hamiltonian chords (m_2-products near the puncture), denoted by $\{a, b, ab, ba, aba, bab, \ldots\}$. One can check that m_1 vanishes. Note that if we consider m_2-operations near the puncture, $m_2(a,a)$, $m_2(b,b)$ vanishes as they are not composable. If $p = 2$ or $q = 2$, we could have an m_2-product coming from a global holomorphic polygon which contributes $m_2(a,a)$ or $m_2(b,b)$ respectively. In general, there are two global J-holomorphic polygons with all of its corners are of word length 1. They are p-gon and q-gon and come from lifts of upper/lower hemisphere of $(M_{F_{p,q}}/G_W)\setminus L)$. Their corners are hamiltonian chords a or b at infinity. They cannot contribute to m_{p-1} or m_{q-1}, only contribute to m_p or m_q respectively. The boundaries of these polygons are whole G_W-orbits of L so they represents the unit element of $CW^{G_W,\bullet}(L,L)$. $\square$

Lemma 7.1.2. *$\widetilde{L}$ split-generate the wrapped Fukaya category of $M_{F_{p,q}}$*

Proof. We proceed as in the work of Heather Lee [Lee16]. To avoid confusion, let us denote by $\tilde{a}$ the sum over G_W orbit of a in this proof. From Abouzaid's generating criterion, it is enough to show that the following open-closed map hits the unit.

$$\mathcal{OC} : CC_\bullet(CW^\bullet(\widetilde{L},\widetilde{L})) \to SH^\bullet(M_{F_{p,q}})$$

We take the following Hochschild cycle

$$\frac{\tilde{a}^{\otimes p}}{p} - \frac{\tilde{b}^{\otimes q}}{q} \in CC_\bullet(CW^\bullet(\widetilde{L},\widetilde{L}))$$

It is not hard to see that $\widetilde{L}$ provides a tessellation of $M_{F_{p,q}}$, which consists of q distinct p-gons and p distinct q-gons. We first check that it is G_W-equivariant Hochschild cycle. From Lemma 7.1.1, it is enough to check m_p, m_q operations respectively.

$$\partial_{Hoch}(\tilde{a}^{\otimes p}/p - \tilde{b}^{\otimes q}/q) = (m_p(\tilde{a},\ldots,\tilde{a}) - m_q(\tilde{b},\ldots,\tilde{b})) = 1_{\widetilde{L}} - 1_{\widetilde{L}} = 0.$$

On the other hand, the image of the open-closed map of this Hochschild cycle is a cocycle represented by the bounded area of $M_{F_{p,q}}$ covering each region with weight one. Note that the orientation of the boundary Lagrangian of p-gon and q-gon are opposite to each other, and thus p-gons and q-gons in the image add up despite the negative sign in the expression $-\tilde{b}^{\otimes q}/q$. □

Let us discuss the mirror matrix factorization. Using localized mirror functor, we can explicitly compute the mirror matrix factorization. Since $\widetilde{W}$ has non-isolated singularity (singularity along z-axis), we need to be a little bit careful in the discussion. Denote by $S = \mathbb{C}[x, y, z]$. By counting appropriate polygons from the picture with sign, we can prove the following lemma, whose proof is left as an exercise.

Lemma 7.1.3. *The localized mirror functor*

$$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_{F_{p,q}}/G_{F_{p,q}}]) \to \mathscr{M}\mathscr{F}(\widetilde{W})$$

sends L to the following matrix factorization

$$M_L = \Big(\; S^{\oplus 2} \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} S^{\oplus 2} \; \Big) \tag{7.1.1}$$

$$\delta_0 = \begin{pmatrix} x & y \\ -y^{p-1} & x^{q-1} + yz \end{pmatrix} \tag{7.1.2}$$

$$\delta_1 = \begin{pmatrix} x^{q-1} + yz & -y \\ y^{p-1} & x \end{pmatrix} \tag{7.1.3}$$

Remark 7.1.4. *If we set $z = 0$, this matrix factorization become a compact generator of $\mathscr{M}\mathscr{F}(W^T)$ corresponding to skyscraper sheaf at the singular point.*

Corollary 7.1.5. *The matrix factorization M_L is of Koszul type. Namely we have an isomorphism*

$$M_L \cong \big(S[\theta_x, \theta_y], \partial_K + \partial'_K\big).$$

Here, θ_1, θ_2 are odd degree generators (hence anti-commute) and

$$\partial_K = x \cdot \iota_{\theta_x} + y \cdot \iota_{\theta_y}, \;\; \partial'_K := W_x \theta_x \wedge \cdot + W_y \theta_y \wedge \cdot$$

where

$$W_x = (x^{q-1} + yz), \quad W_y = y^{p-1}$$

Remark 7.1.6. *The following Koszul complex has cohomology* $\mathbb{C}[z]$ *concentrated on the right end.*

$$K(x,y) := 0 \to S(\theta_1 \wedge \theta_2) \xrightarrow{\partial_K} S\theta_1 \oplus S\theta_2 \xrightarrow{\partial_K} S \to 0$$

Therefore, following Dyckerhoff [Dyc11], we compute its endomorphism algebra $End(M_L)$.

Lemma 7.1.7. $End_{\mathscr{MF}_{dg}}(M_L)$ *is quasi-isomorphic to a DG algebra of polynomial differential operators*

$$\mathrm{Hom}_{\mathscr{MF}_{dg}}(M_L, M_L) \simeq \Big(S[\partial_{\theta_x}, \partial_{\theta_y}, (\theta_x\wedge), (\theta_y\wedge)], D\Big) \tag{7.1.4}$$

$$D(\partial_{\theta_x}) = W_x, \quad D(\partial_{\theta_y}) = W_y \tag{7.1.5}$$

$$D(\theta_x\wedge) = x, \quad D(\theta_y\wedge) = y \tag{7.1.6}$$

Its cohomology is

$$H^\bullet(\mathrm{Hom}_{\mathscr{MF}_{dg}}(M_L, M_L)) \simeq \mathbb{C}[z][\Gamma_x, \Gamma_y] \tag{7.1.7}$$

$$\Gamma_x = [\partial_{\theta_x} - x^{q-2}(\theta_x\wedge) - z(\theta_y\wedge)] \tag{7.1.8}$$

$$\Gamma_y = [\partial_{\theta_y} - y^{p-2}(\theta_y\wedge)] \tag{7.1.9}$$

Proof. The first part of the lemma is obvious because morphisms of a matrix factorization of Koszul type are those of exterior algebras. It is easy to check that the differential satisfies given equations. For example,

$$D(\partial_{\theta_x}) = [\partial_K + \partial'_K, \partial_{\theta_x}] \tag{7.1.10}$$

$$= [\partial'_K, \partial_{\theta_x}] \tag{7.1.11}$$

$$= [(W_x\theta_x\wedge), \iota_{\theta_x}] = W_x \tag{7.1.12}$$

To each differential operators we can assign an order of its symbols. It provides a decreasing filteration $\{F^i\}$ on the complex. The first page of the spectral sequence associated to the filtration is a dual Koszul complex associated to a regular sequence (x, y)

$$E_1 = H^\bullet\Big(K^\vee(x,y) \otimes_{\mathbb{C}} \mathbb{C}[\partial_{\theta_x}, \partial_{\theta_y}], \partial_K^\vee \otimes 1\Big) \simeq \mathbb{C}[z][\partial_{\theta_x}, \partial_{\theta_y}]$$

In particular we know that the cohomology algebra is a $\mathbb{C}[z]$- modules of rank less or equal to 4. On the other hand, the cycles generated by Γ_x, Γ_y in the lemma have already provided four $C[z]$-linear independent element. Therefore the spectral sequence degenerates at E_1 page. This finishes the proof. □

We can show that our mirror functor is fully-faithful.

Lemma 7.1.8. *The first-order part of the mirror functor is*

$$\mathscr{F}_1^{\mathbb{L}} : CW^\bullet(L, L) \to Hom_{\mathscr{MF}}(M_L, M_L) \tag{7.1.13}$$

$$a \to \Gamma_x \tag{7.1.14}$$

$$b \to \Gamma_y. \tag{7.1.15}$$

It is a quasi-isomorphism. Therefore $\mathscr{F}^{\mathbb{L}}$ embedds $\mathscr{W}\mathscr{F}(M_{F_{p,q}})$ as a full subcategory of $\mathscr{MF}(\widetilde{W})$.

Proof. From the Figure 7.1, we see that $\mathscr{F}_1^{\mathbb{L}}$ sends a to Γ_x and b to Γ_y. Moreover,

$$[\Gamma_x, \Gamma_y] = [-z(\theta_y \wedge), \partial_{\theta_y}] = z.$$

Therefore $ab + ba$ hits z and $\mathscr{F}_1^{\mathbb{L}}$ is surjective.

Notice that $CW^\bullet(L, L)$ and $H^\bullet(Hom_{\mathscr{MF}}(M_L, M_L))$ are filtered by

$$F^k := (ab+ba)^k \cdot CW^\bullet(L, L), \quad G^l := z^l \cdot H^\bullet(Hom_{\mathscr{MF}}(M_L, M_L))$$

It is easy to check that $\mathscr{F}_1^{\mathbb{L}}$ is a filtered map with respect to $F^\bullet$ and $G^\bullet$.

The graded piece F^0/F^1 is a 4 dimensional vector space spanned by four words $< 1, a, b, ab >$. This is because

$$aba = (ab+ba) \cdot a - \delta_{2,q} b, \quad bab = (ab+ba) \cdot b - \delta_{2,p} a.$$

An element $ab + ba$ is in the center of the algebra. Therefore

$$F^k/F^{k+1} \simeq (ab+ba)^k \cdot F^0/F^1 = (ab+ba)^k \cdot < 1, a, b, ab > .$$

By a similar reason, we have

$$G^k/G^{k+1} \simeq z^k \cdot G^0/G^1 = z^k \cdot <1, \Gamma_x, \Gamma_y, (\Gamma_x \circ \Gamma_y)>$$

The induced morphism of associated graded $Gr\mathscr{F}_1^{\mathbb{L}}$ is an isomorphism of vector spaces at every level. By the comparison theorem, so is $\mathscr{F}_1^{\mathbb{L}}$. □

Corollary 7.1.9. *$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_{F_{p,q}}/G_{F_{p,q}}]) \to \mathscr{M}\mathscr{F}(W^T + xyz)$ is a quasi-equivalence.*

Proof. It is enough to show that M_L and $M_{\mathbb{L}}$ generates $\mathrm{MF}(W^T + xyz)$. Orlov's equivalence

$$\mathrm{MF}(W^T + xyz) \simeq D_{sg}(W^T + xyz)$$

$$\left(M^1 \underset{\psi}{\overset{\phi}{\rightleftarrows}} M^0 \right) \mapsto Coker(\psi)$$

sends $M_{\mathbb{L}}$ to a skyscraper sheaf $\mathscr{O}_o$ at the orgin and M_L to a structure sheaf $\mathscr{O}_z$ of z-axis. These are two irreducible components of a critical locus of $W^T + xyz$. Therefore it generates $\mathrm{MF}(W^T + xyz)$.(See [Ste13]) □

7.2 Chain cases

The polynomial $W = C_{p,q} = x^p + xy^q$ has maximal symmetry group $G_W = \mathbb{Z}/pq$. We proceed as in the Fermat case. Denote by ξ the following generator of G_W:

$$x \to e^{\frac{2\pi i}{p}} \cdot x, \quad y \to e^{\frac{-2\pi i}{pq}} \cdot y.$$

Recall that the quotient space $[M_{C_{p,q}}/G_W]$ has one orbifold points of order q and two punctures of order pq and $\frac{pq}{gcd(p-1,q)}$, respectively. Let us call them as B_1, B_2 respectively. The orbifold action near B_1 is generated by ξ while the action near B_2 is generated by ξ^{p-1} by Proposition 5.2.1.

We take a Lagrangian L connecting B_1 and B_2 in $\mathbb{P}^1_{pq,q,\frac{pq}{gcd(p-1,q)}}$ (we may take the part of the equator between B_1 and B_2). And denote by $\widetilde{L}$ the sum of all lifts of L in the Milnor fiber.

Lemma 7.2.1. *The wrapped Floer complex $CW^\bullet(L,L)$ is quasi-isomorphic to the following model.*

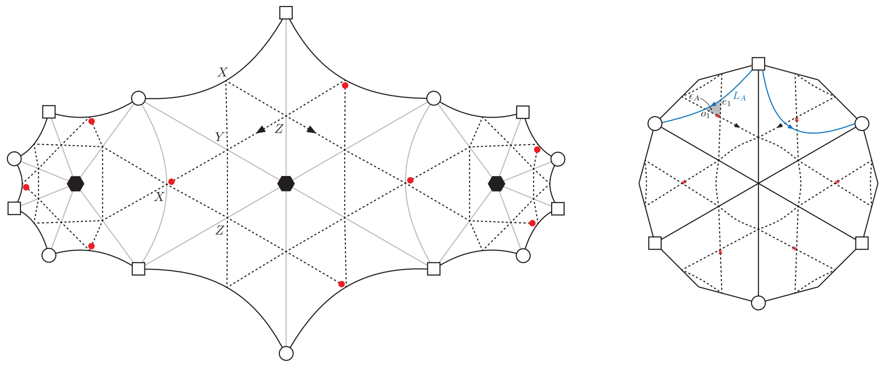


Figure 7.2: Milnor fiber of $E7 = C_{3,3}$ and a choice of Lagrangian L

1. *As a vector space,*
$$CW^\bullet(L,L) \simeq \mathbb{C}[a,b]/(ab=0)$$
Here, a, b are even variables.

2. *m_1 vanishes and m_2 coincides with a polynomial multiplication.*

3. *$m_k(a,b,a,b\ldots) = 0$ for $1 \le k \le 2q-1$ and $m_{2q}(a,b,\ldots,a,b) = 1$. Likewise, $m_k(b,a,b,a\ldots) = 0$ for $1 \le k \le 2q-1$ and $m_{2q}(b,a,\ldots,b,a) = 1$*

Its Ω and H^1-grading is given by the following table.

	1_L	a	b
Ω-*grading*	0	0	2
H^1-*grading*	0	$-\gamma_1$	$-\gamma_3$

Proof. Branches of $\widetilde{L}$ don't intersect with each other in the interior. Therefore $CW^\bullet(L,L)$ consists of hamiltonian chords at infinity near B_1 or B_2. Among them we choose two generators between the nearest orbits. Namely, choose one lift $\widetilde{L}_1$ and take the wrapped generator

- a, the shortest chord $\in CW^\bullet(\widetilde{L}_1, \xi^{-1}\cdot\widetilde{L}_1)$ near B_1
- b, the shortest chord $\in CW^\bullet(\widetilde{L}_1, \xi^{1-p}\cdot\widetilde{L}_1)$ near B_2.

Here, a (resp. b) is nothing but the chord between $\widetilde{L}_1$ and its clockwise rotation at B_1 (resp. B_2). Namely, recall that ξ, ξ^{p-1} correspond to γ_1, γ_3 of the orbifold fundamental group in the Proposition 5.2.1. And $\gamma_1^{-1}, \gamma_3^{-1}$ are the minimal clockwise rotations in the uniformizing neighborhood of orbifold points. Therefore $\xi \cdot \widetilde{L}_1$ is obtained by clockwise rotation of $\widetilde{L}_1$(centered at B_1) sending B_2-vertex to the nearest B_2-vertex. The same holds for $\xi^{p-1} \cdot \widetilde{L}_1$ switching the role of B_1 and B_2.

We can also concatenate them to create new Hamitonian chords, namely $a^2, a^3, \ldots, b^2, b^3, \ldots$. We cannot concatenate different words as their heads and tails are different from each other. The rest of the argument is similar to the Fermat case. m_1 vanishes because there are no J-holomorphic strip between them. Concatenating two chords corresponds to m_2 operation concentrated near the punctures. The first global J-holomorphic polygon contributes to a non-trivial A_∞ operation is a $2q$-gon. It is a lift of an orbifold bigon $(M_{C_{p,q}}/G_W) \setminus L$. It's corners consists of q many a and b alternating each other. □

Lemma 7.2.2. *$\widetilde{L}$ generates the wrapped Fukaya category of $M_{C_{p,q}}$*

Proof. We proceed as in the Fermat case. Milnor fiber $M_{C_{p,q}}$ is tessellated by p copies of $2q$-gons that are considered in the previous lemma. In Figure 7.2, this is given by 3 copies of hexagons. To show that open-closed map hits the unit, we take the following Hochschild cycle.

$$\frac{1}{q}(\tilde{a} \otimes \tilde{b})^{\otimes q} \in CC_\bullet(CW^\bullet(\widetilde{L}, \widetilde{L}))$$

It is indeed a cycle because

$$\partial_{Hoch}(\frac{1}{q}(\tilde{a} \otimes \tilde{b})^{\otimes q}) = m_{2q}(\tilde{a}, \tilde{b}, \ldots, \tilde{a}, \tilde{b}) - m_{2q}(\tilde{b}, \tilde{a}, \ldots, \tilde{b}, \tilde{a}) \tag{7.2.1}$$

$$= 1_{\widetilde{L}} - 1_{\widetilde{L}} = 0 \tag{7.2.2}$$

On the other hand, the image of the open-closed map of this Hochschild cycle is a cocycle represented by the bounded area of $M_{C_{p,q}}$ covering each region with weight one. □

If we solve the weak Maurer-Cartan equation for L, we get the potential $\widetilde{W} = x^q + xyz$, which can be also written as $W^T + xyg$ with $g(x, y, z) = z - y^{p-1}$.

Lemma 7.2.3. *The localized mirror functor*

$$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_{C_{p,q}}/G_{C_{p,q}}]) \to \mathscr{M}\mathscr{F}(\widetilde{W})$$

sends L to the following matrix factorization

$$M_L = (\ S \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} S\ ) \tag{7.2.3}$$

$$\delta_0 = x \tag{7.2.4}$$

$$\delta_1 = x^{q-1} + yz \tag{7.2.5}$$

Proof. This follows from the Figure 7.2. □

The matrix factorization M_L we get is again of Koszul type. It is even simpler; it is an actual factorization of $\tilde{W}$. One can check directly that

$$M_L = \big(S[\theta_x], (x \cdot i_{\theta_x} + W_x \cdot \theta_x \wedge)\big), \quad W_x = x^{q-1} + yz$$

Using the same technique,

Lemma 7.2.4. *The self-hom space of M_L is quasi-isomorphic to a DG algebra of polynomial differential operators*

$$Hom_{\mathrm{MF}}(M_L, M_L) \simeq \Big(S[\partial_{\theta_x}, (\theta_x \wedge)], D\Big) \tag{7.2.6}$$

$$D(\partial_{\theta_x}) = W_x, \tag{7.2.7}$$

$$D(\theta_x \wedge) = x. \tag{7.2.8}$$

Its cohomology is concentrated to even degree, isomorphic to

$$H^\bullet(Hom_{\mathrm{MF}}(M_L, M_L)) \simeq \mathbb{C}[y,z]/(yz = 0)$$

Proof. The first part of the lemma is same as Fermat case. The cohomology computation can be done in a similar way, but we found that it is much easier to do it by hands. This complex is isomorphic to the 2-periodic complex

$$S^{\oplus 2,even} \underset{D_1}{\overset{D_0}{\rightleftarrows}} S^{\oplus 2,odd} \tag{7.2.9}$$

$$D_0 = \begin{pmatrix} x & -x \\ W_x & -W_x \end{pmatrix} \tag{7.2.10}$$

$$D_1 = \begin{pmatrix} x & -W_x \\ x & -W_x \end{pmatrix} \tag{7.2.11}$$

Therefore, we have

$$Ker(D_0)/Im(D_1) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,even} \middle| a = b \right\} \Big/ S \cdot \begin{pmatrix} x \\ x \end{pmatrix} \oplus S \cdot \begin{pmatrix} W_x \\ W_x \end{pmatrix} \tag{7.2.12}$$

$$\simeq \mathbb{C}[x,y,z]/(x = x^{p-1} + yz = 0) \tag{7.2.13}$$

$$\simeq \mathbb{C}[y,z]/(yz = 0) \tag{7.2.14}$$

$$Ker(D_1)/Im(D_0) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,odd} \middle| W_x \cdot a + x \cdot b = 0 \right\} \Big/ S \cdot \begin{pmatrix} x \\ -W_x \end{pmatrix} \tag{7.2.15}$$

$$\simeq 0 \tag{7.2.16}$$

The last equality holds because (x, W_x) is a regular sequence of S. □

Now we can show that our mirror functor is an equivalence.

Lemma 7.2.5. *The first-order part of the mirror functor is given by*

$$\mathscr{F}_1^{\mathbb{L}} : CW^\bullet(L,L) \to Hom_{\mathscr{MF}}(M_L, M_L) \tag{7.2.17}$$

$$a \to y \tag{7.2.18}$$

$$b \to z. \tag{7.2.19}$$

It is a quasi-isomorphism. Moreover $\mathscr{F}^{\mathbb{L}} : \mathscr{WF}([M_{C_{p,q}}/G_{C_{p,q}}]) \to \mathscr{MF}(x^{q-1} + xyz)$ *is a quasi-equivalence.*

Proof. Similar to Fermat case. □

7.3 Loop cases

The loop type polynomial $W = x_1^p x_2 + x_1 x_2^q$ has $G_W = \mathbb{Z}/pq-1$ as the maximal diagonal symmetry group. One notable difference of a loop type from the others is that the action of the maximal diagonal symmetry is free. The quotient $M_{L_{p,q}}/G$ is an honest three punctured sphere. Its wrapped Fukaya category and its homological mirror symmetry was proved in [AAE⁺13]. The result in this section can be essentially found therein, except that we use localized mirror functor to define the explicit correspondences.

Let us introduce more notation. For loop type, we use variables x_i ($i = 1,2,3$) instead of x, y, z. Let ξ denote the following generators of this group.

$$x_1 \to e^{\frac{2q\pi i}{pq-1}} \cdot x_1, \quad x_2 \to e^{\frac{-2\pi i}{pq-1}} \cdot x_2$$

Also recall three puctures are of order $pq-1, pq-1$ and $\frac{pq-1}{gcd(p-1,q-1)}$. Let's denote them by B_1, B_2, B_3 respectively. A cyclic orbifold action is generated by ξ near B_1, by ξ^{-p} near B_2 and by ξ^{1-p} near B_3 by the Proposition 5.2.1. As there are three punctures, we choose three shortest Lagrangians L_i from B_{i+1} to B_{i+2} for $i = 1,2,3 \mod 3$ which are part of the equator sphere passing through 3 punctures. The following can be checked from [AAE⁺13].

Lemma 7.3.1. *The wrapped Floer complexes $CW^\bullet(L_i, L_j)$ is quasi-isomorphic to the following model.*

1. *as a vector space,*

$$CW^\bullet(L_i, L_j) \simeq \begin{cases} \mathbb{C}[a_{i+1}, b_{i+2}]/(a_{i+1}b_{i+2} = 0) & i = j \\ \mathbb{C} < a_i^n \cdot c_{i,j} \cdot b_j^m >, \ n, m \in \mathbb{N} & i \neq j \end{cases}$$

   Here, a_i, b_i are even an and $c_{i,j}$ are odd.

2. *m_1 vanishes and m_2 coincides with a polynomial multiplication and an obvious bimodule structure.*

3. *$m_3(c_{12}, c_{23}, c_{31}) = 1$*

Its Ω and H^1-grading is given by the following table.

	1_{L_i}	a_i	b_{i+1}	c_{12}	c_{23}	c_{31}
Ω-*grading*	0	$2\cdot\delta_{i,3}$	$2\cdot\delta_{i,3}$	-1	1	1
H^1-*grading*	0	$-\gamma_i$	$-\gamma_i$		*not defined*	

Consider the direct sum of lifts $\widetilde{L}_i$ of L_i in $M_{L_{p,q}}$. Then the following is well-known.

Lemma 7.3.2. $\{\widetilde{L}_i\}_{i=1,2,3}$ *split-generates the wrapped Fukaya category of* $M_{L_{p,q}}$

Next, we move on to the mirror computation. For the Seidel Lagrangian $\mathbb{L}$ in the quotient $[M_{L_{p,q}}/G_W]$, the potential function can be computed as

$$\widetilde{W} = xyz.$$

Remark 7.3.3. *We may write*

$$xyz = x^p y + xy^q + xy(z - x^{p-1} - y^{q-1}) = W^t + xy\cdot g(x,y,z)$$

From the picture, it is easy to check the following.

Lemma 7.3.4. *The localized mirror functor*

$$\mathscr{F}^{\mathbb{L}} : \mathscr{WF}([M_{L_{p,q}}/G_{L_{p,q}}]) \to \mathscr{MF}(x_1x_2x_3)$$

sends L to a following matrix factorization

$$M_{L_i} = (\ S \underset{\delta_{i,1}}{\overset{\delta_{i,0}}{\rightleftarrows}} S\ ) \tag{7.3.1}$$

$$\delta_0 = x_i,\quad \delta_1 = \frac{x_1x_2x_3}{x_i} \tag{7.3.2}$$

As before, we can also write this matrix factorization as

$$M_{L_i} = \big(S[\theta_{x_i}], (x_i\cdot i_{\theta_{x_i}} + W_{x_i}\cdot\theta_{x_i}\wedge)\big),\quad W_{x_i} = \frac{x_1x_2x_3}{x_i}.$$

For later purpose, we calculate hom complex by hand. (in [AAE+13] it was proved using Orlov's equivalence.)

Lemma 7.3.5. *The self-hom space of M_{L_i} is quasi-isomorphic to a DG algebra of polynomial differential operators*

$$Hom_{\mathrm{MF}}(M_{L_i}, M_{L_i}) \simeq \left(S[\partial_{\theta_{x_i}}, (\theta_{x_i}\wedge)], D\right) \tag{7.3.3}$$

$$D(\partial_{\theta_x}) = W_{x_i}, \tag{7.3.4}$$

$$D(\theta_x \wedge) = x_i. \tag{7.3.5}$$

The cohomology of Floer complexes are given as follows;

$$H^\bullet(Hom_{\mathrm{MF}}(M_{L_i}, M_{L_i})) \simeq \begin{cases} \mathbb{C}[x_1, x_2, x_3]/(x_i = W_{x_i} = 0) & i = j \\ \mathbb{C}[x_1, x_2, x_3] \cdot (\frac{x_1 x_2 x_3}{x_i x_j})/(x_i = x_j = 0) & i \neq j \end{cases}$$

Proof. A computation of self-Floer complex is almost identical to that of chain type. A complex of morphism $Hom_{\mathrm{MF}}(M_{L_i}, M_{L_j})$ is isomorphic to

$$S^{\oplus 2, even} \underset{D_1}{\overset{D_0}{\rightleftarrows}} S^{\oplus 2, odd} \tag{7.3.6}$$

$$D_0 = \begin{pmatrix} x_i & -x_j \\ W_{x_j} & -W_{x_i} \end{pmatrix} \tag{7.3.7}$$

$$D_1 = \begin{pmatrix} W_{x_i} & -x_j \\ W_{x_j} & -x_i \end{pmatrix} \tag{7.3.8}$$

Therefore, we have

$$Ker(D_0)/Im(D_1) = \begin{cases} \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,even} \middle| a = b \right\} \Big/ S \cdot \begin{pmatrix} x_i \\ x_i \end{pmatrix} + S \cdot \begin{pmatrix} W_{x_i} \\ W_{x_i} \end{pmatrix} & (i = j) \\ \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,even} \middle| x_i \cdot a = x_j \cdot b \right\} \Big/ S \cdot \begin{pmatrix} x_j \\ x_i \end{pmatrix} + S \cdot \begin{pmatrix} W_{x_i} \\ W_{x_j} \end{pmatrix} & (i \neq j) \end{cases} \tag{7.3.9}$$

$$\simeq \begin{cases} \mathbb{C}[x_1, x_2, x_3]/(x_i = W_{x_i} = 0) & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{7.3.10}$$

$$Ker(D_1)/Im(D_0) = \begin{cases} \left\{ \begin{pmatrix} c \\ d \end{pmatrix} \in S^{2,odd} \middle| W_{x_i} \cdot c = x_i \cdot d \right\} \Big/ S \cdot \begin{pmatrix} x_i \\ W_{x_i} \end{pmatrix} & (i = j) \\ \left\{ \begin{pmatrix} c \\ d \end{pmatrix} \in S^{2,odd} \middle| W_{x_i} \cdot c = x_j \cdot d \right\} \Big/ S \cdot \begin{pmatrix} x_i \\ W_{x_j} \end{pmatrix} + S \cdot \begin{pmatrix} x_j \\ W_{x_i} \end{pmatrix} & (i \neq j) \end{cases} \tag{7.3.11}$$

$$\simeq \begin{cases} 0 & (i = j) \\ \mathbb{C}[x_1, x_2, x_3] \cdot (\frac{x_1 x_2 x_3}{x_i x_j})/(x_i = x_j = 0) & (i \neq j) \end{cases} \tag{7.3.12}$$

□

Now we can show that our mirror functor is an equivalence as before.

Lemma 7.3.6. *The first-order part of the mirror functor is*

$$\mathscr{F}_1^{\mathbb{L}} : CW^{\bullet}(L_i, L_j) \to Hom_{\mathscr{MF}}(M_{L_i}, M_{L_j}) \tag{7.3.13}$$

$$a_i \to x_i \tag{7.3.14}$$

$$b_i \to x_i \tag{7.3.15}$$

$$c_{i,j} \to \frac{x_1 x_2 x_3}{x_i x_j} \tag{7.3.16}$$

It is a quasi-isomorphism. Moreover, $\mathscr{F}^{\mathbb{L}} : \mathscr{WF}([M_{C_{p,q}}/G_{L_{p,q}}]) \to \mathscr{MF}(x_1 x_2 x_3)$ *is a quasi-equivalence.*

Chapter 8

New Fukaya category for Landau-Ginzburg orbifolds

For a weighted homogeneous polynomial $W : \mathbb{C}^n \to \mathbb{C}$, let G_W be the maximal diagonal symmetry group which can be defined as in the two variable cases. Landau-Ginzburg orbifold is a pair (W, G') for a choice of subgroup $G' < G_W$. We plan to define a $\mathbb{Z}/2$-graded A_∞-category for (W, G').

8.1 Preliminaries

Definition 8.1.1. *A polynomial W is called weighted homogeneous polynomial if*

$$W(\lambda^{w_1} z_1, \cdots, \lambda^{w_n} z_n) = \lambda^h W(z_1, \cdots, z_n)$$

for $w_1, \cdots, w_n, h \in \mathbb{Z}$. We say W has weight $(w_1, \cdots, w_n; h)$. We will always assume that W has an isolated singularity at the origin. Namely, $\mathrm{grad} W = (\frac{\partial W}{\partial z_1}, \cdots, \frac{\partial W}{\partial z_n})$ *vanishes only at* $0 \in \mathbb{C}^n$.

We set $V_t = V_t(W) = \{z \in \mathbb{C}^n \mid W(z) = t\}$, and V_0 is an hypersurface of isolated singularity at 0 and V_t $(t \neq 0)$ is non-singular. Milnor fiber M_W is nothing but $V_1(W)$. For the well-known Milnor fibration

$$\frac{W}{|W|} : S_\epsilon^{2n-1} \setminus K \to S^1$$

with $K = (S^{2n-1}_\epsilon \cap V_0)$, its fiber is diffeomorphic to M_W. Geometric monodromy $h : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$h(x_1, \cdots, x_m) = (e^{2\pi i w_1/h} x_1, \cdots, e^{2\pi i w_m/h} x_m) \tag{8.1.1}$$

which restricts to $h : M_W \to M_W$. It is known that $S^{2n-1}_\epsilon \setminus K$ is diffeomorphic to the manifold obtained by identifying two ends of $M_W \times [0,1]$ by h. (see [Mil68] Lemma 9.4)

One can define its closure $\overline{M}_W$ and its boundary $\partial\overline{M}_W$. There are monodromy homomorphism (from a parallel transport fixing the boundary)

$$h_* : H_*(\overline{M}_W) \to H_*(\overline{M}_W), h_* : H_*(\overline{M}_W, \partial\overline{M}_W) \to H_*(\overline{M}_W, \partial\overline{M}_W) \tag{8.1.2}$$

A topological precursor of our construction is a **variation operator** (around the origin in $\mathbb{C}$)

$$\mathrm{var} : H_{n-1}(\overline{M}_W, \partial\overline{M}_W) \to H_*(\overline{M}_W). \tag{8.1.3}$$

It is defined by sending $[c] \to (h_* - id)([c])$.

We want to find a symplectic categorical analogue of this variation operator for weighted homogenous polynomials. At first, we will define a distinguished Reeb orbit Γ_W from the geometric monodromy (8.1.1). The analogue of monodromy homomorphism (8.1.2) will be the quantum cap action by Γ_W.

$$\cap\Gamma_W : \mathcal{WF}(L,L) \to \mathcal{WF}(L,L).$$

Then, the analogue of the variation operator (8.1.3) will be an assignment

$$L \mapsto \mathrm{Cone}\left(\ L \xrightarrow{\cap\Gamma_W} L \ \right)$$

Recall that in Floer theory, it is well-known that taking a Lagrangian surgery corresponds to taking a cone complex. Roughly speaking, we are taking a surgery of non-compact Lagrangians for the Reeb chords at infinity to turn it into a compact object, namely the corresponding vanishing cycle.

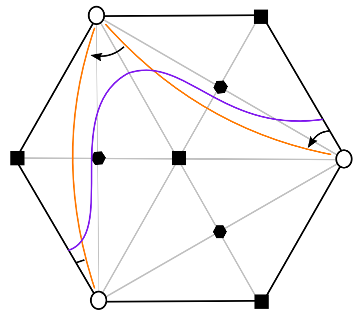


Figure 8.1: Lagrangian L, monodromy action and vanishing cycle

8.2 Monodromy, Reeb orbit, and $\mathscr{C}_{\Gamma_W}$

Let W be a n-variable weighted homogeneous polynomial of weight $(w_1,\dots,w_n;h)$. Consider a quadratic hamltonian

$$H := \frac{1}{2h}\sum_1^n w_i \cdot |x_i|^2$$

which generates a circle action of $\mathbb{C}^n$ of a given weight. The Hamiltonian flow $\Phi_W(s)$ of X_H is called *monodromy flow*.

Definition 8.2.1. *A monodromy transformation* $\Phi_W = \Phi_W(1)$ *is a time*-1 *hamiltonian flow of* H.

$$x_i \mapsto e^{\frac{2\pi i w_i}{h}} x_i.$$

Geometrically, a hamiltonian action of H is a lifting of a rotation action of a base of a fibration $W : \mathbb{C}^K \to \mathbb{C}$. Therefore a time 1 flow restricts to each fiber. More precisely, W satisfies

$$W(t^{w_1}\cdot x_1,\dots t^{w_n}\cdot x_n) = t^h W(x_1,\dots,x_n).$$

In particular, we have

$$W(e^{\frac{2\pi i w_1}{h}s}\cdot x_1,\dots e^{\frac{2\pi i w_n}{h}s}\cdot x_n) = e^{2\pi i s}$$

which means that the flow of H acts as a circle action of an S^1 family of Milnor fiber $W = e^{2\pi i s}$. Set $s = 1$ then we get a desired automorphism. Furthermore, the monodromy flow restricts to a singular fiber $W^{-1}(0)$ and its boundary link

$$L_{W,\delta} := W^{-1}(0) \cap S^{2n-1}_{\delta}.$$

We choose symplectic form ω and a Liouville form λ on $\mathbb{C}^k$ as

$$\omega = \sum_k \frac{1}{2\pi i w_k} dz_k \wedge d\overline{z}_k, \quad \lambda = \sum_k \frac{i}{4\pi w_k}(z_k d\overline{z}_k - \overline{z}_k dz_k). \tag{8.2.1}$$

Then the monodromy flow $\Phi_W(s)$ becomes a Reeb flow $\mathscr{R}$ on $L_{W,\delta}$, where the contact one form is given by a restriction of λ. Starting from any point $x \in L_{W,\delta}$, we get a Reeb chords

$$\gamma : [0,1] \to \partial_\infty W^{-1}(0), \quad \gamma(0) = x, \quad \gamma(1) = \Phi(x)$$

This is an **orbits** of a free quotient $\left(L_{W,\delta}/\langle\Phi\rangle\right)$. Therefore a space of time-1 Reeb orbits of the quotient is a total space $L_{W,\delta}/\langle\Phi\rangle$.

Although we haven't defined full-fledged orbifold symplectic cochains in general, the idea of [CFHW96] [Sei06a] and [KvK16] still works. The ciritical set of action fuctional is a total space $L_{W,\delta}/\langle\Phi\rangle$. A local Floer cohomology $CF^{\bullet}_{loc}(L_{W,\delta}/\langle\Phi\rangle, H)$ is isomorphic to its Morse cohomology. Notice that the Reeb flow we are using is complex linear on the nose. No non-trivial local system needs to be introduced.

Definition 8.2.2. *A Reeb orbit $\Gamma_W \in CF^{\bullet}_{loc}(L_{W,\delta}/\langle\Phi\rangle, H)$ is defined to be a cocyle corresponds to a fundamental class*

$$\Gamma_W \leftrightarrow \left[L_{W,\delta}/\langle\Phi\rangle\right] \in H^{\bullet}\left(L_{W,\delta}/\langle\Phi\rangle; \mathbb{Z}\right).$$

Although the fibration $W : \mathbb{C}^n \to \mathbb{C}$ is singular at the origin, its restriction

$$W|_{S^{2n-1}} : S^{2n-1} \to \mathbb{C}$$

does not. We can canonically identify $L_{W,\delta}$ and $\partial M_{W,cpt} \times \{1\} \subset M_W$. The Reeb orbit Γ_W becomes an hamiltonian orbit, still denoted by Γ_W, of a quotient orbifold $W^{-1}(1)/\langle\Phi\rangle$. Notice that a Φ is always an element of a maximal symmetry group G_W. Therefore, we get an analogous Hamiltonian orbits Γ_W of $[M_W/G_W]$.

It is still possible to check that Γ_W is "closed" in a suitable sense.

Lemma 8.2.3. *Suppose H_{S^1} is G-equivariant, $H_{S^1} > 0$, and C^2-small Morse perturbation of H inside a compact region. Then there is no smooth pseudo-holomorphic cylinder satisfying*

$$u : S^1 \times \mathbb{R} \to [M_W / G_W] \quad \lim_{s\to\infty} u(t,s) = \Gamma_W(t).$$

whose output

$$\gamma_-(t) := \lim_{s\to-\infty} u(s,t)$$

does not lie in $(L_{W,\delta}/G_W) \times \{1\}$.

Proof. At first, we can rule out the case when the output is outside a compact region using the idea of a spectral sequence [Sei06a] associated to an action filtration. We explain what is going on. For a non-trivial orbit $\gamma \in \mathcal{O}(H_{S^1})$ at the end, its action is given by

$$A_{H_{S^1}}(\gamma) := -\int_{S^1} \gamma^* \lambda + \int_0^1 H_{S^1}(\gamma(t))dt \tag{8.2.2}$$

$$= -2\int_0^1 r^2 dt + \int_0^1 H(\gamma(t))dt + \int_0^1 F(\gamma(t))dt, \quad (H_{S^1} = H(r) + F(r,t)) \tag{8.2.3}$$

$$= -\int_0^1 r^2 + \epsilon \quad (\epsilon << 1) \tag{8.2.4}$$

Nontrivial Hamiltonian orbits are appears as a small perturbation of orbits of level n, which means that it is a perturbation of Hamiltonian orbits $\gamma' \subset (L_{W,\delta}/G_W) \times \{n\}$ of H. An action value of such orbit is dominated by $-n^2$. The orbit $\Gamma_W(t)$ is an orbit of level 1.

Since $H_{S^1} > 0$, a topological energy of u

$$E_{top}(u) := \int_{S^1\times\mathbb{R}} \omega - d(u^* H_{S^1} \cdot dt) = A_{H_{S^1}}(\gamma_-) - A_{H_{S^1}}(\Gamma_W)$$

must be positive. Therefore, the output γ_- cannot be an orbit of level $n \geq 2$.

Suppose γ_- is an orbit inside a compact region, a Morse critical point of H. since u provides a homotopy class of orbifold loops, we must have $\Phi(\gamma_-(0)) =$

$\gamma_-(1)$. It means that γ_- must be a fixed point of a monodromy Γ. It is impossible because the only fixed point of Γ is the origin, which does not contained in M_W. □

A degeneration of pseudo-holomorphic curves involving Γ_W may have a cylinder breaking, but we conclude that they always comes in cancelling pairs, and hence does not appear in the equations. In particular, an operation $m^{\Gamma}_{n,F,\phi}$ on $\mathscr{W}\mathscr{F}([M_W/G_W])$ can be used to define the desired A_∞-structure.

Definition 8.2.4. *A Fukaya category of a Landau-Ginzburg pair* (W, G_W) *is defined to be*

$$\mathscr{F}(W, G_W) := \mathscr{C}_{\Gamma_W}.$$

For general LG orbifold, we make the following definition (cf. Berglund-Henningson [BH95], Seidel [Sei15]).

Definition 8.2.5. *For any subgroup* $G' < G_W$*, we define* $(G')^T = Hom(G_W/G', \mathbb{C}^*)$*. We define the* $\mathbb{Z}/2$*-graded Fukaya category of the pair* (W, G') *to be the semi-direct product.*

$$\mathscr{F}(W, G') := \mathscr{C}_{\Gamma_W} \rtimes (G')^T$$

For the maximal group $G' = G_W$, the Fukaya category is the same as the one constructed above.

$$\mathscr{F}(W, G_W) = \mathscr{C}_{\Gamma_W}.$$

We will see in the next section that for invertible curve singularities, the Mirror of the monodromy action is given by the restriction of LG model to a hypersurface in Theorem 9.2.1, and expected to hold in general dimensions

Chapter 9

Berglund-Hübsch HMS for curve singularity

In this section, we finally state and prove homological mirror symmetry for Berglund-Hübsch pairs of invertible curve singularities.

Let W be one of the invertible curve singularities. In the previous section, we have defined new A_∞-category $\mathscr{C}_{\Gamma_W}$ using wrapped Fukaya category of the Milnor fiber of W, and quantum cap action of monodromy orbit Γ_W. We first prove

Theorem 9.0.1. *There is a geometric A_∞-functor*

$$\mathscr{G}^{\mathbb{L}} : \mathscr{F}(W, G_W) \to \mathscr{MF}(W^T)$$

which gives a derived equivalence.

This also proves the full version of homological mirror symmetry between Berglund-Hübsch pairs.

Corollary 9.0.2 (Berglund-Hübsch HMS)**.** *For any subgroup $G' < G$, we have the following derive equivalence of $\mathbb{Z}/2$-graded categories*

$$\mathscr{F}(W, G') \cong \mathscr{MF}^{(G')^T}(W^T)$$

where the latter is $(G')^T$-equivariant matrix factorization category of W^T.

The corollary can be deduced from the main theorem, as both sides can be written as semi-direct products, and the functor can be shown to be equivariant ([CHL17]).

A proof of Theorem 9.0.1 occupies the rest of the section. Recall that we have constructed the following HMS for Milnor fibers using localized mirror functor $\mathscr{F}^{\mathbb{L}}$ in Theorem 7.0.1.

$$\mathscr{W}\mathscr{F}([M_W/G_W]) \xrightarrow{\mathscr{F}^{\mathbb{L}}} \mathrm{MF}(W^T + xyg) \ ,$$

where a polynomial g was given by

$$\mathrm{Fermat}: g = z, \ \ \mathrm{Chain}: g = (z - y^{q-1}), \mathrm{Loop}: g = (z - x^{p-1} - y^{q-1}).$$

The proof consists of the following two theorems. At first, we compute the class Γ_W explicitly and show that it is a mirror to $\cdot g$.

Theorem 9.0.3. *There is a diagram of A_∞-bimodules*

$$\begin{tikzcd}
\mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r, "\cap\Gamma_W"] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{F}(W, G_W) \arrow[r] \arrow[d, "\widetilde{\mathscr{F}}^{\mathbb{L}}"] & {} \\
\mathrm{MF}(W^T + xyg) \arrow[r, "\cdot g"] & \mathrm{MF}(W^T + xyg) \arrow[r] & \mathrm{MF}(W^T) \arrow[r] & {}
\end{tikzcd}$$

whose all vertical lines are quasi-isomorphisms.

Next, we enhance this equivalence to that of A_∞ categories.

Theorem 9.0.4. *There exist an A_∞-functor, $\mathscr{G}^{\mathbb{L}}$, extending the bimodule map $\widetilde{\mathscr{F}}^{\mathbb{L}}$*

$$\mathscr{G}^{\mathbb{L}} : \mathscr{F}(W, G_W) \to \mathscr{M}\mathscr{F}(W^T)$$

which gives a derived equivalence.

9.1 Computation of Γ_W

We compute Γ_W for invertible curve singularities. The boundary $\partial M_{W,cpt}$ is a union of circle and G_W acts on them by rotation. In this particular case, Γ_W is represented by a union of loops around punctures.

Proposition 9.1.1. *A class Γ_W is given by a sum of hamiltonian orbits, geometrically represented by the following element of $\pi_1\left(\mathbb{P}^1_{a,b,c}\right)$ respectively.*

1. ***Fermat type*** $\left(\simeq \mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}\right)$*: $\Gamma_W \leftrightarrow (\gamma_3)^{-1}$;*

2. ***Chain type*** $\left(\simeq \mathbb{P}^1_{pq,q,\frac{pq}{gcd(p-1,q)}}\right)$*: $\Gamma_W \leftrightarrow (\gamma_1)^{1-p} + (\gamma_3)^{-1}$;*

3. ***Loop type*** $\left(\simeq \mathbb{P}^1_{pq-1,pq-1,\frac{pq}{gcd(p-1,q-1)}}\right)$*: $\Gamma_W \leftrightarrow (\gamma_1)^{1-p} + (\gamma_2)^{1-q} + (\gamma_3)^{-1}$.*

Proof. The idea is that locally around zero, $W^{-1}(c)$ shares the same coordinate system near the punctures. We will describe an orbit of an induce circle action around the puncture. Those orbits are tranversally nondegenerate, appears as an S^1 family of orbit. Γ_W corresponds to a fundamental class of such S^1-family.

1. **Fermat type** $x^p + y^q$
   Recall that the weight of this polynomial is $(pq; q, p)$ so the Reeb flow is
   $$(x, y) \mapsto (e^{\frac{2\pi i t}{p}} x, e^{\frac{2\pi i t}{q}} y).$$
   The quotient M_W/G has a single puncture at infinity. Under a coordinate change
   $$X = \frac{1}{x}, \quad Y = \frac{1}{y}$$
   the equation for $W^{-1}(c)$ becomes $X^p + Y^q = cX^pY^q$. It is a Riemann surface of a mutivalued function
   $$Y = \left(\frac{X^p}{cX^p - 1}\right)^{\frac{1}{q}}.$$

The only brach point we should consider is that of 0. A local coordinate chart near the puncture is given by

$$w^{\frac{q}{gcd(p,q)}} = X = \frac{1}{x}$$

and the induced flow is given by

$$w \mapsto e^{-\frac{2\cdot gcd(p,q)\pi i t}{pq}} \cdot w$$

Its winding number around ∞ is -1.

2. **Chain type** $x^p + xy^q$
The weight of this polynomial is $(q, p-1, pq)$ and Reeb flow is

$$(x, y) \mapsto (e^{\frac{2\pi i t}{p}} x, e^{\frac{2(p-1)\pi i t}{pq}} y).$$

The quotient M_W/G_W has two punctures at 0 and and ∞. Around $x = 0$, $W^{-1}(c)$ is a Riemann surface of a function

$$\frac{1}{y} = Y = \left(\frac{x}{c - x^p}\right)^{\frac{1}{q}}$$

with branch points 0 and $(\xi_p)^k c^{\frac{1}{p}}$ where ξ_p is a p-th root of unity. Notice that brach points are converging to 0 as $c \to 0$. We want to find a local chart for $c \to 0$, so we choose lines connecting 0 and $(\xi_p)^k c^{\frac{1}{p}}$ $(k \neq 0)$ as branch cuts. Consider a small loop inside an x-plane encircling 0 and c both. It sends $(x, y) \mapsto (x, e^{\frac{2q\pi i}{p-1}} y)$. Therefore y^{-1} is a function of $x^{\frac{1-p}{q}}$ locally around $x = 0$. A local coordinate chart near 0 and induced flow will be

$$(w_1)^q = x^{1-p}, \quad w_1 \mapsto e^{\frac{-2(p-1)\pi i t}{pq}}$$

The winding number of a corresponding time-1 orbit is $(1 - p)$. At ∞, we can do the same thing as we did for Fermat type. Under a coordinate change

$$X = \frac{1}{x}, \quad Y = \frac{1}{y}$$

the equation for $W^{-1}(c)$ becomes $X^{p-1} + Y^q = cX^pY^q$. It is a Riemann surface of a mutivalued function

$$Y = \left(\frac{X^{p-1}}{cX^p - 1}\right)^{\frac{1}{q}}.$$

Therefore, a local coordinate chart near the puncture and induced flow becomes

$$(w_2)^{\frac{q}{gcd(p-1,q)}} = X = \frac{1}{x}, \quad w_2 \mapsto e^{\frac{-2\cdot gcd(p-1,q)\pi i t}{pq}} \cdot w_2$$

A winding number around ∞ is -1.

3. **Loop type** $x^py + xy^q$
The weight of this polynomial is $(q-1, p-1, pq-1)$. Reeb flow is given by

$$(x, y) \mapsto (e^{\frac{2(q-1)\pi i t}{pq-1}} x, e^{\frac{2(p-1)\pi i t}{pq-1}} y).$$

The quotient M_W/G_W has three punctures at 0 ,1 and ∞. The following reparametrization presents $W^{-1}(c)$ as Riemann surface of functions of z.

$$x = \left(\frac{z^q}{c-z}\right)^{\frac{1}{pq-1}}, \quad y = \left(\frac{(c-z)^p}{z}\right)^{\frac{1}{pq-1}}$$

To find a local coordinate near $x = 0$ when $c \to 0$, choose a line connecting 0 and c inside a z-plane as a branch cut. Again, consider a small loop inside a z-plane around 0 and c both. It sends $(x, y) \mapsto (e^{\frac{2(q-1)\pi i}{pq-1}} x, e^{\frac{2(p-1)\pi i}{pq-1}} y)$. Therefore y^{-1} is a function of $x^{\frac{1-p}{q-1}}$ locally around $x = 0$. Therefore, a local coordinate and induced flow will be

$$(w_1)^{q-1} = x^{1-p}, \quad w_1 \mapsto e^{\frac{2(1-p)\pi i t}{pq-1}}$$

The winding number of a corresponding time-1 orbit is $(1-p)$. The rest of the procedure is entirely the same.

□

9.2 Mirror of the Monodromy action: Restriction of LG model to a hypersurface

On the symplectic side, we will consider the monodromy Γ_W-action(quantum cap action in 3.3.3) to define the new A_∞-category $\mathscr{C}_{\Gamma_W}$, and on the complex side, we will consider the restriction to the hypersurface $g(x,y,z)=0$.

Proposition 9.2.1. *The following diagram*

$$\begin{tikzcd} \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r, "\cap\Gamma_W"] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[d, "\mathscr{F}^{\mathbb{L}}"] \\ \mathrm{MF}(\widetilde{W}) \arrow[r, "g"] & \mathrm{MF}(\widetilde{W}) \end{tikzcd}$$

commutes up to homotopy H. More precisely, we have pre-homomorphism of A_∞-bimodules $H^{\mathbb{L}}=\{H^{\mathbb{L}}_k\}_{k=1}^{\infty}$, satisfying

$$(H^{\mathbb{L}}\circ m \pm D\circ H^{\mathbb{L}})(a_1,\ldots,\underline{b},\ldots,a_n) \tag{9.2.1}$$
$$= \sum_{j\le k, j+l\ge k} \mathscr{F}^{\mathbb{L}}_{n-l+1}(a_1,\ldots,\cap\Gamma_W(a_{j+1},\ldots,\underline{b},\ldots,a_{j+l}),\ldots,a_n) \tag{9.2.2}$$
$$\pm g\cdot(\mathscr{F}^{\mathbb{L}}_{n+1}(a_1,\ldots,\underline{b},\ldots,a_n)) \tag{9.2.3}$$

Proof. Recall $\boldsymbol{b}=xX=yY+zZ$ denotes a bounding cochain of $\mathbb{L}$. Here, we use bold font for $\boldsymbol{b}$ in this section to emphasize its role, and also denote the component of cap action by $(\cap\Gamma_W)_l=N_l$ for simplicity.

Definition 9.2.2. *Define a pre-homomorphism of bimodules $H^{\mathbb{L}}=\{H^{\mathbb{L}}_k\}_{k=1}^{\infty}$ as*

$$H^{\mathbb{L}}_{n+1}(a_1,\ldots,a_k,\underline{b},a_{k+1},\ldots,a_n)(x)=\sum_l N_{n+l+2}(\mathbf{b},\ldots,\mathbf{b},x,a_1,\ldots,\underline{b},\ldots,a_n)$$

Also define

$$\mathscr{H}^{\mathbb{L}}_0(x)=\sum_l N_{l+1}(\mathbf{b},\ldots,\mathbf{b},\underline{x}),\quad x\in CW(\mathbb{L},L_1)$$

Notice that $\mathscr{H}^{\mathbb{L}}_0$ is not a part of a bimodule map. Consider a boundary of 1-dimensional components of moduli space governing $\mathscr{H}^{\mathbb{L}}(a_1,\ldots,\underline{b},\ldots,a_n)$. There are four different possible degeneration

- (string of $\boldsymbol{b}$s is not broken, the interior point remains) This components contribute to

$$\sum_{j\le k, j+l\ge k} \mathscr{F}^{\mathbb{L}}_{n-l+1}(a_1,\dots,N_{l+1}(a_{j+1},\dots,\underline{b},\dots,a_{j+l}),\dots,a_n).$$

- (string of $\boldsymbol{b}$s is not broken, the interior point escapes towards special output) This components contribute to

$$H^{\mathbb{L}} \circ m(a_1,\dots,\underline{b},\dots,a_n) = \sum H^{\mathbb{L}}_{n-l+1}(a_1,\dots,m_{l+1}(a_{j+1},\dots,\dots,a_{j+l}),\dots,a_n)$$

  Notice that since the interior point escaped toward the special output, the special input $\underline{b}$ is no more special w.r.t m_k operation. it can be anywhere.

- (string of $\boldsymbol{b}$s is broken, the interior point remains) This components contributes to

$$D \circ H^{\mathbb{L}} = [m_1^{\mathbf{b}}, H^{\mathbb{L}}] \text{ and } \widetilde{W}\cdot H^{\mathbb{L}}(a_1,\dots,\underline{b},\dots,a_n)$$

- (string of $\boldsymbol{b}$s is broken, the interior point escapes toward special output) This components contributes to

$$\mathscr{H}^{\mathbb{L}}_0 \cdot (\mathscr{F}^{\mathbb{L}}_{n+1}(a_1,\dots,\underline{b},\dots,a_n))$$

Notice that in the category of matrix factorization, the element $(\widetilde{W})$ acts as a zero. Therefore, to prove 9.2.1, we have to show

$$\mathscr{H}^{\mathbb{L}}_0(\alpha) = g\cdot\alpha, \quad ,\forall \alpha \in Hom(\mathbb{L}, L), \quad \forall L \in Ob(\mathscr{W}\mathscr{F}([M_W/G_W]))$$

For this, we need to recall Kodaira-Spencer map, which is special case of the closed-open map. Closed-open map refers to a map from closed string theory to open string theory, and more concretely, for a closed (resp. open) symplectic manifold M, the following maps are expected to be ring isomorphisms.

$$QH^\bullet(M) \to HH^\bullet(\mathscr{F}(M))\big(\text{resp. } SH^\bullet(M) \to HH^\bullet(\mathscr{W}\mathscr{F}(M))\big)$$

We first recall the known results related to our construction. Fukaya-Oh-Ohta-Ono [FOOO16] constructed a Kodaira-Spencer map $QH^*(M) \to Jac(W)$ by counting holomorphic discs with interior insertion of a quantum cohomology class with Lagrangian boundary condition. Such construction was generalized to $\mathbb{P}^1_{a,b,c}$ by the first author with Amorim, Hong and Lau [ACHL20], which we will adapt to our cases at hand.

When the output image of a closed-open map is a multiple $c[L]$ of fundamental class of Lagrangian, this coefficient c suitably decorated with deformation variables provide such a map. For a Liouville domain M, closed-open map from symplectic cohomology was first introduced by Seidel [Sei06b], and it plays a crucial role in Abouzaid's work on generation criterion of wrapped Fukaya category [Abo10]. The first of these map is given by

$$CO_0 : SH^\bullet(M) \to HW^\bullet(L,L)$$

Pascaleff [Pas19] proved that for the complement of normal crossing anti-canonical divisor, and for a Lagrangian section L, CO_0 gives isomorphism in degree zero. On the other hand, Tonkonog [Ton19] found quite interesting relationship between potential functions of Lagrangians on Fano manifolds and the symplectic cohomology ring of the smooth anti-canonical complement.

In our case, we apply these ideas to the Seidel Lagrangian $\mathbb{L}$ in the quotient $[M_W/G_W]$.

Definition 9.2.3. *Kodaira-Spencer map*

$$KS^{\mathbf{b}} : SH^{even}([M_W/G_W]) \to \mathrm{Jac}(\widetilde{W})$$

is defined by the reading the coefficient of $[\mathbb{L}]$ *of the output of closed-open map given by an interior insertion of a symplectic cohomology class and boundary insertion of* $\mathbf{b}$*'s.*

The fact that this map is well-defined can be proved as in [ACHL20], and we omit the details.

Proposition 9.2.4. *Up to homotopy,*

$$\mathcal{H}_0^{\mathbb{L}}(\alpha) = KS^{\mathbf{b}}(\Gamma_W) \circ \alpha.$$

Proof. Consider a moduli space of pseudo-holomorphic curves governing $\mathscr{H}_0^{\mathbb{L}}$, but we allow its interior point to move towards boundary on $\mathbb{L}$. An operation associated to the moduli space provides a homotopy between $\mathscr{H}_0^{\mathbb{L}}(\alpha)$ and $\mathrm{KS}^{\mathbf{b}}(\Gamma)\circ\alpha$. See Figure 9.1. $\square$

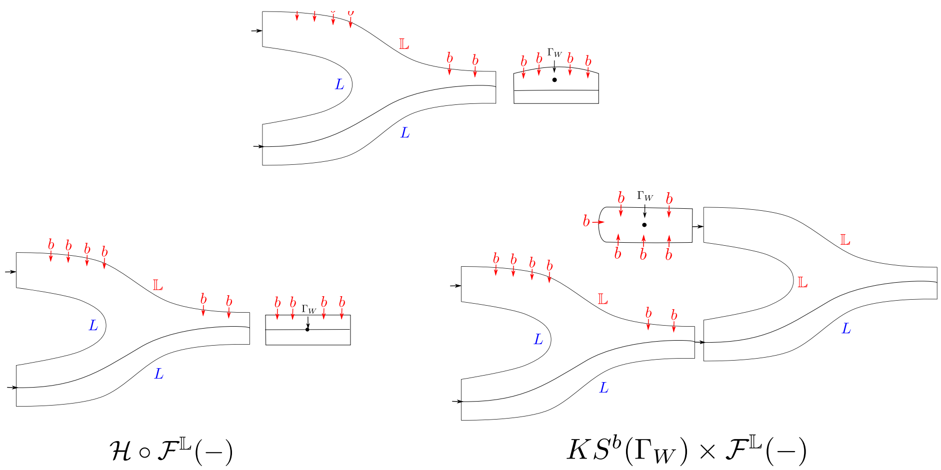


Figure 9.1: Kodaira-Spencer invariant appears.

Proposition 9.2.5.

$$KS^{\mathbf{b}}(\Gamma_W) = g\cdot 1$$

Proof. We should count pseudo-holomorphic discs whose boundary lies in Seidel's Lagrangian $\mathbb{L}$, corners are at immersed intersections of $\mathbb{L}$ and interior puncture asymptotic to a Hamiltonian orbit Γ_W. The $\Omega-$ and H^1-grading of $SH^\bullet$ and $CF^\bullet(\mathbb{L},\mathbb{L})$ must be compatible. It implies that the only possible contribution of $\mathrm{KS}^{\mathbf{b}}(\gamma_i^k)$ is a k-th power of a variable associated to the immersed corner of $\mathbb{L}$ opposite to the puncture. We can see such a polygon in a picture explicitly as in Figure 9.2. A careful sign computation combined with 9.1.1 will calculates $\mathrm{KS}^{\mathbf{b}}(\Gamma)$ explicitly.

1. Fermat type $F_{p,q}$: $\mathrm{KS}^{\mathbf{b}}(\Gamma) = z\cdot 1$

2. Chain type $C_{p,q}$: $\mathrm{KS}^{\mathbf{b}}(\Gamma) = (z-y^{p-1})\cdot 1$

3. Loop type $L_{p,q}$: $\mathrm{KS}^{\mathbf{b}}(\Gamma) = (z - x^{q-1} - y^{p-1}) \cdot 1$

These polynomials are exactly $g(x, y, z)$ we want.

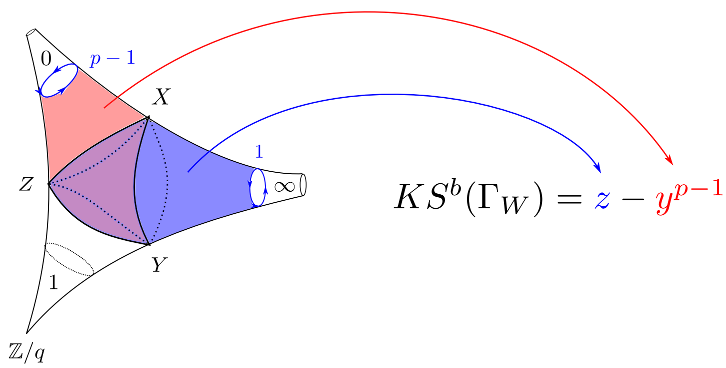


Figure 9.2: $\mathrm{KS}^{\mathbf{b}}(\Gamma_W)$ for a chain type singularity

To prove that there are no other contribution, we use the idea of Tonkonog [Ton19] of domain stretching. Namely, if we consider a compactification of $[M_W/G_W]$ into $\mathbb{P}^1_{a,b,c}$, holomorphic discs that contribute to the closed-open map from $QH^\bullet(\mathbb{P}^1_{a,b,c}) \to Jac(\widetilde{W})$ has been worked out in [ACHL20]. If we interpret Reeb orbits as suitable orbifold insertions, we obtain the above computations. We can relate it to the computation of $CO_0(\Gamma) \in \mathrm{Jac}(\widetilde{W})$ using the construction of Tonkonog. For curve singularities, the hypersurface Σ in [Ton19] is just a point (or points) playing the role of Donaldson hypersurface.

Given a holomorphic disc with boundary on a compact Lagrangian K in the compact space X, Tonkonog introduced a new stretching procedure for holomorphic curves based on domain stretching as in standard Floer theory. By clever choice of sequence of Hamiltonians (called S-shaped) for the domain stretching, the standard J-holomorphic discs breaks into parts which share Reeb orbits as same asymptotics. Reeb orbits for S-shaped Hamiltonian are divided into types I, II, III, IV_a, IV_b, depending on their position with respect to Liouville collar. Key part of the proof is to show that only type II Reeb orbit appears in the breaking. In our case, if breaking occurs at type I, IV_a, IV_b Reeb orbits (which are constant orbits), then collecting the parts from this constant orbit to Σ we get a non-trivial sphere that maps to X. The starting polygon in X do not intersect other vertices of $\mathbb{P}^1_{a,b,c}$, and this intersection number with perturbed J-holomorphic curve is positive, so the sphere should not intersect other vertices. Therefore, such sphere cannot exist. This excludes these type of Reeb orbits as breaking orbits. The argument again type III orbit using no escape lemma still applies to our case. One can see that any disc bubble would increase the intersection with vertices of orbisphere, hence do not occur.

□

The proof of 9.2.1 is now complete. □

We obtain the Theorem 9.0.3 as a corollary.

Corollary 9.2.6. *The following diagram commutes up to homotopy;*

$$\begin{tikzcd}
\mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r, "\cap\Gamma_W"] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{F}(W, G_W) \arrow[r] \arrow[d, "\widetilde{\mathscr{F}}^{\mathbb{L}}"] & {} \\
\mathrm{MF}(W^T + xyg) \arrow[r, "\cdot g"] & \mathrm{MF}(W^T + xyg) \arrow[r] & \mathrm{MF}(W^T) \arrow[r] & {}
\end{tikzcd}$$

where $\widetilde{\mathscr{F}}^{\mathbb{L}} = \begin{pmatrix} \mathscr{F}^{\mathbb{L}} & \mathscr{H}^{\mathbb{L}} \\ 0 & \mathscr{F}^{\mathbb{L}} \end{pmatrix}$. *Each row is a distinguished triangle of bimodules. All vertical lines induces quasi-isomorphisms.*

This establish an equivalence of $\mathscr{F}(W, G_W)$ and $\mathrm{MF}(W^T)$ at the level of bimodules. To state a full mirror symmetry statement, we are going to promote it to an equivalence of A_∞ category.

9.3 Berglund-Hübsch mirror symmetry

We introduce a relevant operation which generalizes $\mathscr{H}_0^{\mathbb{L}}$ in the previous section.

Definition 9.3.1. *Define $\mathscr{H}^{\mathbb{L}}$ as follows;*

$$\mathscr{H}^{\mathbb{L}}_k : \mathrm{Hom}_{\mathscr{F}(W,G_W)}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{F}(W,G_W)}(L_{k-1}, L_k) \to \mathrm{Hom}(M_{L_0}, M_{L_k}).$$

$$\mathscr{H}^{\mathbb{L}}_k(a_1, \ldots, \epsilon b_{i_1}, \ldots, \epsilon b_{i_j}, \ldots, a_k)(x) = \sum_{l,p} \pm m^{\Gamma_W}_{l+k+2,\{(l+1)+\widehat{F}^p\}\cup(l+1)}(\boldsymbol{b}, \ldots, \boldsymbol{b}, x, a_1, \ldots, a_k)$$

Here, $F = \{i_1, \ldots, i_j\}$ and $(l+1) + \widehat{F}^p$ means a translation of $\widehat{F}^p$ by $(l+1)$.

Definition 9.3.2. *Let*

$$\mathscr{G}^{\mathbb{L}} : \mathscr{F}(W, G_W) \to \mathrm{MF}(W^T)$$

as a pre-A_∞ functor

$$\mathscr{G}^{\mathbb{L}} = M^{\mathbf{b}} + \mathscr{H}^{\mathbb{L}}_k$$

More precisely, it is defined as

$$\mathscr{G}^{\mathbb{L}}(L) = \mathrm{Cone}\left(\mathscr{F}^{\mathbb{L}}(L) \xrightarrow{g} \mathscr{F}^{\mathbb{L}}(L) \right)$$

and

$$\begin{aligned}
&\left[\mathscr{G}^{\mathbb{L}}_k(a_1, \ldots, \epsilon b_{i_1}, \ldots, \epsilon b_{i_j}, \ldots, a_k)\right](x) \\
&= \sum_{l} m^{\Gamma_W}_{k+l+1,(l+1)+F}(\boldsymbol{b}, \ldots, \boldsymbol{b}, x, a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, a_k) \\
&+ \sum_{l,p} m^{\Gamma_W}_{l+k+2,\{(l+1)+\widehat{F}^p\}\cup(l+1)}(\boldsymbol{b}, \ldots, \boldsymbol{b}, x, a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, a_k) \\
&+ \epsilon \cdot \sum_{l,p} m^{\Gamma_W}_{k+l+1,(l+1)+\widehat{F}^p}(\boldsymbol{b}, \ldots, \boldsymbol{b}, x, a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, a_k)
\end{aligned}$$

Proposition 9.3.3. *$\mathscr{G}^{\mathbb{L}}$ is an A_∞ functor.*

Proof. The caculation is similar. Consider an 1-dimensinal components of moduli space governing

$$\sum_{l} m^{\Gamma_W}_{k+l+1,(l+1)+F}(\boldsymbol{b}, \ldots, \boldsymbol{b}, x, a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, a_k).$$

There are two different possible degenerations.

- (string of $\boldsymbol{b}$s is not broken) This components contribute to

$$\sum_{l,1\le q\le r\le k,F_i} m^{\Gamma_W}_{k+l-(r-q),(l+1)+F_1}(\boldsymbol{b},\dots,\boldsymbol{b},x,a_1,\dots,b_{i_1},\dots$$

$$\dots,m^{\Gamma_W}_{r-q+1,(l+1)+F_2}(a_q,\dots,a_r)\dots,b_{i_j},\dots,a_k).$$

  Here, F_i are possible admissible cuts of F. It corresponds to a

$$\mathscr{G}^{\mathbb{L}}(a_1,\dots,M(\dots),\dots,a_k).$$

- (string of $\mathbf{b}$s is broken) This components contribute to

$$\sum_{l,l_1+l_2=l,q\le k,F_i} m^{\Gamma_W}_{k+l_1+1-r,(l+1)+F_1}(\boldsymbol{b},\dots,\boldsymbol{b},m^{\Gamma_W}_{l_2+r+1,(l_1+1)+F_2}(\boldsymbol{b}\dots,\boldsymbol{b},x,a_1,\dots$$

$$\dots,b_{i_1},\dots,a_q,\dots,a_r)\dots,b_{i_j},\dots,a_k).$$

  Here, F_i are possible admissible cuts of F. It corresponds to a

$$\mathscr{G}^{\mathbb{L}}(\dots)\circ\widetilde{\mathscr{G}}(\dots) \text{ or } [m_1^{\boldsymbol{b}},\mathscr{G}^{\mathbb{L}}(\dots)].$$

$\square$

Finally, we get

Corollary 9.3.4. *The functor*

$$\mathscr{G}^{\mathbb{L}}:\mathscr{F}(W,G_W)\to\mathscr{MF}(W^T)$$

is an A_∞ equivalence which fits into a diagram

$$\begin{array}{ccccccc}
\mathscr{WF}([M_W/G_W]) & \xrightarrow{\cap\Gamma_W} & \mathscr{WF}([M_W/G_W]) & \longrightarrow & \mathscr{F}(W,G_W) & \longrightarrow \\
\downarrow{\scriptstyle\mathscr{F}^{\mathbb{L}}} & & \downarrow{\scriptstyle\mathscr{F}^{\mathbb{L}}} & & \downarrow{\scriptstyle\mathscr{G}^{\mathbb{L}}} & \\
\mathrm{MF}(W^T+xyg) & \xrightarrow{g} & \mathrm{MF}(W^T+xyg) & \longrightarrow & \mathrm{MF}(W^T) & \longrightarrow
\end{array}$$

Proof. Equivalence statement follows directly from a bimodule level computation we did in the last section. Next, simply notice that the first-order component of $\mathscr{G}^{\mathbb{L}}$ coincides with $\widetilde{\mathscr{F}}^{\mathbb{L}}$ is the last subsection. $\square$

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국문초록

이 논문에서는 리우빌 다양체 M의 사교 코호몰로지 군의 원소 $\Gamma \in SH^\bullet(M)$ 가 주어져 있을 때, Γ 의 양자 곱 작용 (quatum cap action) $\Gamma : CW^\bullet(L,L) \to CW^\bullet(L,L)$ 이 호모토피적으로 사라지는 새로운 호모토피 결합 범주 (A_∞-category) $\mathscr{C}_\Gamma$ 를 건설하고자 한다.

이 새로운 건설법을 바탕으로 하여 가중 동차 다항식 W과 그것의 대칭군 G로 이루어진 사교 란다우-긴즈버그(Landau-Ginzburg) 모델 (W,G) 을 만든다. 밀너 올(MIlnor fiber) 의 감긴 푸카야 범주 (wrapped Fukya category) 와 그것에 작용하는 모노드로미 작용 (monodromy action) 을 사용하여, 모노드로미 작용이 사라지는 새로운 범주 $\mathscr{F}(W,G)$를 만든다. 이것은 고전적인 특이점 이론의 변분 연산자(variation operator)의 사교기하적 유추로 간주할 수 있다.

이에 더해, 모노드로미 작용의 거울 현상이 거울 란다우-긴즈버그 모델을 특정한 초곡면에 제한시키는 것임을 보인다. 그것의 응용으로, 모든 가역 곡선 특이점에 대해 버글룬드-흌스 추측을 증명한다.

주요어휘: 라그랑지언 플로어 이론, 거울대칭, 오비폴드, 가역 다항식, 행렬 인수분해

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이학박사 학위논문

Fukaya category for Landau-Ginzburg orbifolds and Berglund-Hübsch conjecture for invertible curve singularities

(란다우-긴즈버그 오비폴드의 푸카야 카테고리와 곡선 가역 특이점의 버글룬드-흅스 추측)

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좌 동 욱

Fukaya category for Landau-Ginzburg orbifolds and Berglund-Hübsch conjecture for invertible curve singularities

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
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by

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Abstract

Fukaya category for Landau-Ginzburg orbifolds and Berglund-Hübsch conjecture for invertible curve singularities

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From a fixed cohomology class $\Gamma \in SH^\bullet(M)$ of a Liouville manifold M, we construct a new A_∞ category denoted by $\mathscr{C}_\Gamma$ on which the quantum cap action of $\Gamma : CW^\bullet(L,L) \to CW^\bullet(L,L)$ vanishes homotopically.

With this construction on one hand, we consider a symplectic Landau - Ginzburg model (W,G) defined by a weighted homogeneous polynomial W and its symmetry group G. From wrapped Fukaya category and a monodromy information of the Milnor fiber, we construct a new Fukaya category $\mathscr{F}(W,G)$ for each pair (W,G) on which the monodromy action vanishes. It is a symplectic analogue of the variation operator in singularity theory.

We also show that the mirror of the monodromy action is a restriction of a mirror Landau-Ginzburg model to a certain hypersurface. As an application, we prove Berglund-Hübsch homological mirror symmetry for all invertible curve singularities.

Key words: Lagrangian Floer theory, Mirror symmetry, Orbifold, Invertible polynomials, Matrix factorization
Student Number: 2012-20261

Contents

Chapter 1

Introduction

Given a singularity W which is a polynomial in $\mathbb{C}^n$, Fukaya-Seidel category is defined by perturbing W into a Morse function W_ϵ and considering the collection of vanishing cycles of W_ϵ and their directed Fukaya A_∞-category. This has been one of the central topic in symplectic geometry and mirror symmetry. In particular, Fukaya-Seidel category defines a symplectic category for a Landau-Ginzburg model W in the setting of homological mirror symmetry conjecture. Namely, if W has a mirror complex manifold M, then Fukaya-Seidel category should be derived equivalent to the derived category of coherent sheaves on M. If W has a mirror Landau-Ginzburg orbifold $(\widehat{W}, H)$ in the sense that $\widehat{W}$ is H-invariant for a finite group H, then FS category should be derived equivalent to maximally graded category of matrix factorization of $\widehat{W}$. Many instances of such homological mirror symmetry has been proved.

In spite of its importance, Fukaya-Seidel category for Landau-Ginzburg orbifold has not been known. The main difficulty is that when a finite group G acts on $\mathbb{C}^n$ and W is G-invariant, its perturbation to a Morse function W_ϵ destroys the original symmetry G. Namely, W_ϵ is not G-invariant in general, and it has not been known how to overcome this difficulty.

In this paper, we introduce a different approach to define a Fukaya category of the singularity W when W is a weighted homogeneous polynomial. In this approach, we will not perturb W and hence we can define an equivariant version of it as well. Namely, for a diagonal symmetry group G of W, we are able to define a new Fukaya category for a Landau-Ginzburg orbifold (W, G).

Instead of using vanishing cycles(which needs perturbation), we will use wrapped Fukaya category of Milnor fiber and monodromy map. Let us recall standard classical singularity theory for analogy. Given an isolated singularity $W : \mathbb{C}^n \to \mathbb{C}$ at the origin, a monodromy map in classical singularity theory is defined by considering the parallel transports along a circle centered at the origin in $\mathbb{C}$. In particular, for the Milnor fiber $X = W^{-1}(1)$, there is a variation map $\mathrm{var} : H_{n-1}(X, \partial X) \to H_{n-1}(X)$ given by the difference of the cycle and its image under monodromy. The image of variation map for Morse singularity are vanishing cycles. Our approach is to use the Milnor fiber and monodromy information to define a Fukaya category, instead of vanishing cycles.

Symplectic cohomology is a version of Hamiltonian Floer cohomology for Liouville domains introduced by Cieliebak, Floer and Hofer [CFH95] and Viterbo [Vit99]

With this new definition of Fukaya category for a Landau-Ginzburg orbifold (W,G), we formulate and prove homological mirror symmetry between invertible curve singularities, called Berglund-Hübsch HMS conjecture.

Berglund-Hübsch introduced mirror pairs for invertible singularities. $W : \mathbb{C}^n \to \mathbb{C}$ is called invertible singularity if W has n-terms and its $n \times n$ exponent matrix E is non-degenerate. Let G be a diagonal symmetry group of W, and let G_W be the maximal diagonal symmetry group of W, which is a finite abelian group. Then, Berglund-Hübsch dual of (W,G) is given by (W^T, G^T) where W^T is another invertible singularity with exponent matrix E^T and $G^T = \mathrm{Hom}(G_W/G, U(1))$. If G is trivial, G^T becomes G_{W^T}, the maximal diagonal symmetry group for the mirror singularity.

The following version of Berglund-Hübsch HMS conjecture has been proved.

$$\text{Fukaya Seidel category of } W \longleftrightarrow MF^{\text{max. gr}}(W^T)$$

In this paper, we prove the following complete form of Berglund-Hübsch HMS conjecture

$$\text{Fukaya category of } (W,G) \longleftrightarrow MF(W^T, G^T)$$

Our proof is constructive and geometric in the sense that we start with (W,G) and we obtain the mirror pair (W^t, G^t) via Floer theory of (W,G), and homological mirror symmetry A_∞-functor is also constructed geometrically.

When $G = G_W$ is maximal, G^T is trivial group and hence right hand side of the above Berglund-Hübsch HMS is a $\mathbb{Z}/2$-graded matrix factorization category of W^T.

The ring $S := \mathbb{C}[x_1, \cdots, x_n]/(W^T)$ is a Cohen-Macaulay ring, and maximal Cohen-Macaulay modules of S has been studied intensively in the 80's. In particular, Eisenbud showed that maximal Cohen-Macaulay modules are equivalent to $\mathbb{Z}/2$-graded matrix factorizations [Eis80].

When W^T is ADE singularity, it is known that there are only finitely many indecomposable objects, and irreducible morphisms between them, and such an information is recorded in the Auslander-Reiten quiver.

Our work provides a geometric interpretation of such Auslander-Reiten quiver. Namely, in the above BH HMS correspondence, we specify non-compact Lagrangians that are mapped to each indecomposable object. Moreover, exact sequences in the Auslander-Reiten quiver can be interpreted as a surgery exact sequence between Lagrangian submanifolds.

Kreuzer-Sharke [KS92] classified invertible singularities in n-variables and have shown that they are given by Thom-Sebastiani sums of Fermat, Chain and Loop type singularities. Thus we may consider the following three families.

- (Fermat type) $F_{p,q} = x^p + y^q$
- (Chain type) $C_{p,q} = x^p + xy^q$
- (Loop type) $L_{p,q} = x^p y + xy^q$

The way that mirror polynomial W^T arise from Floer theory of (W, G_W) is quite interesting. One may consider the mirror of the Milnor fiber M_W of W, its maximal symmetry group G_W, and Fukaya category of an orbifold $[M_W/G_W]$ without considering the monodromy map. In this case, we find that we have $\widetilde{W} : \mathbb{C}^3 \to \mathbb{C}$, which are of the form

$$x^p + y^q + xyz, y^q + xyz, xyz$$

for Fermat, Chain, Loop cases, which is not the transpose mirror polynomial W^T.

But if we restrict $\widetilde{W}$ to a certain graph hypersurface $g(x,y,z) = 0$ then we obtain the transpose polynomial W^T. Namely, if we set

$$z = 0, z - x^{p-1} = 0, z - x^{p-1} - y^{q-1} = 0.$$

we obtain the mirror polynomials as expected for every cases:

$$x^p + y^q, x^p y + y^q, x^p y + x y^q.$$

We find that this restriction $g(x,y,z) = 0$ comes from the monodromy information of the singularity. Namely, there is a distinguished degree zero symplectic cohomology class Γ_W, which come from family of Reeb chords on a quotient orbifold of a Milnor fiber $[\partial M_W / G_W]$. These Reeb chords are exactly the monodromy map around the origin for weighted homogeneous polynomials. We prove that closed-open string map from symplectic cohomology of $[M_W/G_W]$ to Jacobian ring of $\widetilde{W}$ exists and it maps Γ to $g(x,y,z)$.

We show that the natural functor from $\mathscr{W}([M_W/G_W])$ to the new A_∞-category $\mathscr{C}_\Gamma$ is mirror to the natural restriction map from $MF(\widetilde{W})$ to $MF(W^T)$. Furthermore, we construct an A_∞-functor from $\mathscr{C}$ to $MF(W^T)$ and show that this is an A_∞-quasi-isomorphism. This proves the Berglund-Hübsch HMS conjecture for the case of maximal diagonal symmetry group (W, G_W) for the A-side. For a subgroup $G \subset G_W$, such an A_∞-functor can be lifted to an equivariant version, proving the rest of the Berglund-Hübsch HMS conjecture.

Chapter 2

Basic Floer theory

In this section, we describe a general Floer theory.

2.1 Liouville manifold with cylindrical end

Our basic object of study will be a Liouville manifold.

Definition 2.1.1. *A Liouville manifold is a symplectic manifold (M,ω) with a one form λ called Liouville form, such that*

$$d\lambda = \omega.$$

The Liouville vector field Z is a symplectic dual of λ.

$$i_Z\omega = \lambda$$

A Liouville manifold M is to have a cylindrical end if there is a compact submanifold $M_{cpt} \subset M$ such that M is the symplectization of M_{cpt}

$$M = M_{cpt} \bigcup_{\partial M_{cpt}} \partial M_{cpt} \times [1,\infty).$$

A subset $\partial M_{cpt} \times [1,\infty]$ is called a (cylindrical) end of M. The flow Z should be *transverse to* ∂M_{cpt} *and it is of the form*

$$Z = r\frac{\partial}{\partial r}.$$

at the cylindrical end.

To simplify the rest of the discussion, we will assume that our Liouville manifold with cylindrical end M have real dimension $2n$ and satisfies

$$c_1(TM) = 0.$$

We will denote

$$\psi^t$$

as a Liouville flow of time $\log t$.

It is easy to check that the restriction $\Lambda = \lambda|_{M_{cpt}}$ is a contact form and the Liouville one form at the end is its rescaling

$$\lambda = r\Lambda$$

Then an automorphism $\phi : M \to M$ of such manifold is given by a automorphism of the contact manifold times identity at the end. A Reeb vector field R is defined by

$$R \in ker(d\Lambda), \quad \Lambda(R) = 1.$$

The following restrictions on Hamiltonians and almost complex structures are standard.

- We will work with a function $H \in C^\infty(M, \mathbb{R})$ such that $H > 0$, C^2-small on M_{cpt} and *quadratic at infinity*;

  $$H(x,r) = ar^2 + b \ \ (a > 0), \quad r \text{ is a coordinate of } [1,\infty)$$

  We denote the class of such function by $\mathcal{H}(M)$

- Whenever we consider a time dependent perturbation $H_{S^1} = H + F : S^1 \times M \to \mathbb{R}$, we assume $H_{S^1} > 0$, C^2-small on M_{cpt} so that the time-1 periodic orbit of H_{S^1} are non-degenerate. This is true for a generic perturbation.

- an almost complex structure J is called *c-rescaled contact type* if

$$-\frac{c}{r}\lambda \circ J = dr$$

  at the end. We denote a class of such almost complex structure by $\mathcal{J}_c(M)$.

Next, we describe a Lagrangian submanifold we want to use. It is a collection $\mathcal{W}$ of exact properly embedded Lagrangian submanifolds, which may not be compact, but satisfies;

$$\text{Liouville one form } \lambda \text{ vanishes on } L \cap \partial M_{cpt} \times [1,\infty).$$

It means that the intersection $L \cap \partial M_{cpt}$ is a Legenderian submanifold, and L is **conical at the end**, i.e

$$L = \left(L \cap M_{cpt}\right) \bigcup \partial\left(L \cap M_{cpt}\right) \times [1,\infty).$$

Furthermore, all such L is required to have vanishing relative first Chern class $2c_1(M,L)$. We attach a spin structure and a grading function on each L. All Lagrangian submanifold we consider will implicitly carry these extra data.

2.2 Degree and index of Hamiltonian orbits and chords

Fix a small, time dependent perturbation $H_{S^1} : S^1 \times M \to \mathbb{R}$ of H. Let

$$\mathcal{O} := \mathcal{O}(M, H_{S^1})$$

be a set of time-1 orbits of S^1-dependent hamiltonian function H_{S^1}. We may assume all orbits are nondegenerate, which is true for a generic choice of F. For each $\gamma \in \mathcal{O}$, we trivialize $\gamma^* TM$ so that it induces the same homotopy class of existing trivial bundle $\gamma^* K_M$. Then, the derivative of a hamiltonian flow $d\phi_{H_{S^1}}$ restricted on γ induces a path of symplectic matrix Φ_t. Let B_t be a path of symmetric matrix which satisfies a differential equation

$$\frac{d}{dt}\Phi_t = J \cdot B_t \cdot \Phi_t.$$

Here, J is a standard complex structure of $\mathbb{C}^n$. Equip $\mathbb{C}$ a negative cylindrical coordinates

$$\mathbb{R}\times S^1 \to \mathbb{C} \tag{2.2.1}$$
$$(s,t) \mapsto e^{-s-2\pi i t}. \tag{2.2.2}$$

Fix any map $B \in C^{\infty}(\mathbb{C}, Mat_{n\times n}(\mathbb{C}))$ such that

$$B(s,t) = J\cdot B_t$$

for $s << 0$. Now define an operator

$$D_\Phi : W^{1,p}\left(\mathbb{C},\mathbb{C}^n\right) \to L^p\left(\mathbb{C},\mathbb{C}^n\right) \tag{2.2.3}$$
$$D_\Phi(X) = \partial_s X + J\cdot\partial_t X + B_t X \tag{2.2.4}$$

This is Fredholm because we have assumed that γ is nondegenerate.

Definition 2.2.1. *An orientation line o_γ associated to a hamiltonian orbits γ is defined as a determinant line of a Fredholm operator;*

$$\mathrm{Det} D_\Phi = \mathrm{Det}(\mathrm{Ker D_\Phi}) \otimes \mathrm{Det}(\mathrm{Coker D_\Phi})^\vee.$$

A degree of o_γ is defined as an index

$$\deg o_\gamma := \mathrm{ind D_\Phi} = \dim_{\mathbb{R}}\mathrm{Ker D_\Phi} - \dim_{\mathbb{R}}\mathrm{Coker D_\Phi}$$

This integer is also called cohomological Conley-Zehnder index in [A+ 12].

An index can be computed topologically in the following way. A *crossing* $t\in(0,1)$ is a time when $\mathrm{Det}(\Phi_t - \mathrm{Id}) = 0$. A *Robbin-Salamon index* of γ is defined as

$$\mu_{RS}(\gamma) = \frac{1}{2}Sgn(B_0) + \sum_{t\in \text{crossing}} Sgn(B_t) + \frac{1}{2}Sgn(B_1).$$

It is known that

$$\deg o_\gamma = n - \mu_{RS}(\gamma).$$

We move on to a hamiltonian chords. For $L_0, L_1 \in \mathscr{W}$, we define

$$\chi(L_0, L_1; H)$$

to be the set of time-1 hamiltonian chords of H from L_0 to L_1. Let's trivialize a^*TM so that it induces a same relative homotopy class of an existing trivial bundle s^*K_M. Then, a chord $x \in \chi(L_0, L_1)$ can be thought as a path of symplectic matrix Ψ_t with Lagrangian boundary conditions TL_i respectively. We obtain a path of symplectic matrix, still denoted by B_t, in a similar fashion. Now, equip $\mathbb{H}$ the following parametrization

$$(-\infty, 0] \times [0,1] \to \mathbb{H} \tag{2.2.5}$$
$$(s,t) \mapsto e^{-\pi s - 2\pi i t + \pi i}. \tag{2.2.6}$$

Also, choose a family of Lagrangian subspaces F_t such that $F_{s\times\{0\}} = TL_0$ and $F_{s\times\{1\}} = TL_1$. It is uniquely defined (up to homotopy) since our Lagrangian submanifold has gradings. Fix any map $B \in C^\infty(\mathbb{H}, Mat_{n\times n}(\mathbb{C}))$ such that

$$B(s,t) = J \cdot B_t$$

for $s << 0$. Now define an operator

$$D_\Psi : W^{1,p}\left(\mathbb{H}, \mathbb{C}^n, F_t\right) \to L^p\left(\mathbb{H}, \mathbb{C}^n\right) \tag{2.2.7}$$
$$D_\Psi(X) = \partial_s X + J \cdot \partial_t X + B_t X \tag{2.2.8}$$

This is Fredholm because we have assumed that a is nondegenerate.

Definition 2.2.2. *An orientation line o_x associated to a hamiltonian chords x is defined as a determinant line of a Fredholm operator;*

$$\mathrm{Det} D_\Psi = \mathrm{Det}(\mathrm{KerD}_\Psi) \otimes \mathrm{Det}(\mathrm{CokerD}_\Psi)^\vee.$$

A degree of o_x is defined as an index

$$\deg o_x := \mathrm{indD}_\Psi = \dim_{\mathbb{R}} \mathrm{KerD}_\Psi - \dim_{\mathbb{R}} \mathrm{CokerD}_\Psi$$

This integer is called Maslov index of x, and it coincides with the Maslov index

$\mu_M(x)$ of Lagrangian path x.

2.3 Moduli space of pseudo-holomorphic curves

We briefly describe a perturbation scheme for general moduli spaces of pseudo-holomorphic curves we use. We refer [Sei08], [AS10], [A$^+$12] and [Gan13] from which most of the material has been borrowed.

Let

$$S_{m_1,m_2;n_1,n_2}$$

be a moduli space of holomorphic discs D with m_1 positive interior markings, m_2 negative interior markings, n_1 positive boundary markings, and n_2 negative boundary marking. In this paper, we will only consider when there is only one positive markings.

$$S_{m;n,1} := S_{m,0;n,1}, \text{ or } S_{m,1;n} := S_{m,1;n,0}$$

The Deligne-Mumford compactification of this moduli space is denoted by $\overline{S}_{m_1,m_2;n_1,n_2}$ Let us denote

- $Z_+ = [0,\infty) \times [0,1]$ with coordinate (s,t)
- $Z_- = (-\infty,0] \times [0,1]$ with coordinate (s,t)
- $C_+ = [0,\infty) \times S^1$ with coordinate (s,t)
- $C_- = (-\infty,0] \times S^1$ with coordinate (s,t).

Definition 2.3.1. *A collection of strip and cylinder data for $S \in S_{m;n,1}$ or $S \in S_{m,1;n}$ is a choice of*

- *Strip-like ends $\epsilon^k_\pm : Z_\pm \to S$ which models a boundary marking x_k*
- *cylindrical ends $\delta^l_\pm : C_\pm \to S$ which models an interior marking y_k*

Such collection is said to be weighted if each strip and cylinder is endowed with a positive real number

- *$w^\pm_{S,k}$ for each strip-like end $\epsilon^k_\pm$*

- $v^{\pm}_{S,l}$ *for each strip-like end* $\delta^l_{\pm}$

Definition 2.3.2. *Let* $(S,\mathfrak{S})$ *denote a holomorphic disc* S *with a collection of weighted strip and cylinder data* $\{\kappa\}$ *with weight* $\{\nu_\kappa\}$.

1. *A one-form* α_S *is said to be compatible to* $\mathfrak{S}$,

$$\kappa^*\alpha_S = \nu_\kappa dt, \quad \forall\kappa$$

2. *An* $\mathfrak{S}$*-adapted rescaling function is a function* $a_S : S \to [1,\infty)$ *such that*

$$\kappa^* a_S = \nu_\kappa, \quad \forall\kappa$$

3. *For a fixed hamiltonian* $H \in \mathscr{H}(M)$, *an* $S-$*dependent Hamiltonian* H_S *is said to be compatible with* $(S,\mathfrak{S},H)$ *if*

$$\kappa^* H_S = \frac{H\circ\psi^{\nu_\kappa}}{\nu_\kappa^2}, \quad \forall\kappa$$

4. *For a fixed* S^1*-dependent almost complex structure* J_t, *an* $S-$*dependent almost complex structure* J_S *is called* $(S,\mathfrak{S},a_S,J_t)$ *- adpted if the following two conditions are satisfied*

$$J_p \in \mathscr{J}_{a_S(p)}, \quad \forall p\in S$$

$$\kappa^* J_S = \left(\phi^{\nu_\kappa}\right)^* J_t, \quad \forall\kappa$$

Finally, we define

Definition 2.3.3. *For a fixed disc S, a Floer data* F_S *consists of*

1. *A collection of weighted strip and cylinder data* $\mathfrak{S}$*;*
2. *a one form* α_S *compatible with* $(S,\mathfrak{S})$ *which is sub-closed, i.e,*

$$d\alpha_S \le 0$$

3. *An* $(S,\mathfrak{S})-$*adapted rescaling function* a_S*;*

4. *An S-dpendent, $(S, \mathfrak{S}, H)$-compatible hamiltonian H_S;*

5. *An S-dependent, $(S, \mathfrak{S}, a_S, J_t)$-adapted almost complex structure J_S.*

Also, we say F_S^1 and F_S^2 are conformally equivalent if F_S^2 is a rescaling by Liouville flow of F_S^1, up to constant ambiguity in the Hamiltonian terms.

A universal and consistent choice of Floer data is a choice of Floer data F_S for all $S \in S_{m;n,1}$ or $S \in S_{m,1;n}$ which varies smoothly over the moduli space. Since the space of Floer data is contractible, we can extend it to $\overline{S}_{m;n,1}$ or $\overline{S}_{m,1;n}$.

Example 2.3.4. *In the simplest case of a strip $S \in S_{0,1,1}$ or a cylinder $S \in S_{1,1;0}$, we choose a canonical strip-like/cylindrical end with weights 1 for all ends. A form* dt *is a compatible sub-closed one form.*

Definition 2.3.5. *Let $\gamma_i \in \mathcal{O}$ be an time-1 Hamiltonian orbits and $a_j \in \chi(L_{j-1}, L_j)$, $j = 1, \dots, n$ and $a_0 \in \chi(L_n, L_0)$ be Hamiltonian chords. Define*

$$\overline{\mathcal{M}}_{m;n,1}(\gamma_1, \dots, \gamma_m; a_1, \dots, a_n, a_0) \tag{2.3.1}$$

a space of maps

$$\left\{u : S \to M : S \in \overline{S}_{m;n,1}\right\} \tag{2.3.2}$$

satisfying the inhomogeneous Cauchy-Riemann equation with respect to J_S

$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{2.3.3}$$

and the following asymptotic/boundary conditions;

$$\lim_{s \to \infty} u \circ \epsilon_-^k(s, \cdot) = a_k \tag{2.3.4}$$

$$\lim_{s \to -\infty} u \circ \epsilon_+^0(s, \cdot) = a_0 \tag{2.3.5}$$

$$\lim_{s \to \infty} u \circ \delta_+^l(s, \cdot) = \gamma_l \tag{2.3.6}$$

$$u(z) \in \psi^{a_S(z)} L_i, \quad z \in \partial_i S, \text{ an } i\text{-th boundary component of } S. \tag{2.3.7}$$

We define $\overline{\mathcal{M}}_{m,1;n}(\gamma_1, \dots, \gamma_m; \gamma_0; a_1, \dots, a_n)$ in a similar fashion.

Here, we implicitly use that Liouville flow induces a $1-1$ correspondence between hamiltonian chords

$$\chi(L_0, L_1; H) \simeq \chi\left(\phi^t L_0, \phi^t L_1; (\phi^t)^* \left(\frac{H}{t}\right)\right).$$

The following compactness and transversality result is standard.

Lemma 2.3.6. *For a generic choice of universal and consistent Floer data,*

1. *The moduli spaces* $\overline{\mathcal{M}}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ *are compact.*
2. *For a given input* $\gamma_i, \quad i = 1, \ldots, m$ *and* $a_j, \quad j = 1, \ldots, n$*, there are only finitely many* a_0 *for which* $\overline{\mathcal{M}}_{m;n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ *is non-empty .*
3. *It is a manifold of dimension*

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}_{m;n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$$

$$= (2m + n - 2) + \deg o_{a_0} - \sum_{i=1}^{m} \deg o_{\gamma_i} - \sum_{j=1}^{n} \deg o_{a_j}$$

Similar result holds for $\overline{\mathcal{M}}_{m,1;n}(\gamma_1, \ldots, \gamma_m, \gamma_0; a_1, \ldots, a_n)$

Proof. See [Gan13]. For a compactness result, one need to assure that the energy of pseudo-holomorphic curves are a priori bounded in M. This estimate is carefully done therein. Transversality result is a standard application of Sard-Smale argument. The dimension formula is also a standard application of Atiyah-Singer index theorem on a linearized Fredholm operator. □

When $\mathcal{M}_{m;n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ has dimension zero so that it is rigid, then a map $u : S \to M$ in that moduli space is isolated. An orientation of the moduli space provides a canonical isomorphism

$$Q_u : \bigotimes_{i=1}^{m} o_{\gamma_i} \otimes \bigotimes_{j=1}^{n} o_{a_j} \to o_{a_0}.$$

We sum up Q_u for all $u \in \mathcal{M}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ and all a_0 and define

$$\mathbf{F}_{m;,n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n)$$

$$:= \sum_{\dim_{\mathbb{R}} \mathcal{M}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n,a_0)=0} \sum_{u \in \mathcal{M}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n,a_0)} Q_u \left(\bigotimes_{i=1}^{m} o_{\gamma_i} \otimes \bigotimes_{j=1}^{n} o_{a_j} \right)$$

We define $\mathbf{F}_{m,1;n}(\gamma_1,\ldots,\gamma_m;a_1,\ldots,a_n)$ in a similar way.

2.4 Wrapped Fukaya category

In this section and the next, we recall a definition of wrapped Fukaya category and symplectic cohomology in a quadratic hamiltonian setup . See [Rit13] or [Gan13] for more detailed discussion. For two Lagrangian submanifolds $L_0, L_1 \in \mathcal{W}$, a *wrapped Floer cochain complex* is a vector space

$$CW^\bullet(L_0, L_1; H) = \bigoplus_{a \in \chi(L_0,L_1;H)} o_a$$

It is graded by the degree $\deg o_a$. We will use the notation a instead of o_a for generators if it cause no confusion.

Definition 2.4.1. *A wrapped Fukaya category $\mathcal{W}\mathcal{F}(M)$ consists of*

1. *a set of objects $\mathcal{W}$*
2. *a space of morphisms $CW^\bullet(L_0, L_1)$ for $L_i \in \mathcal{W}$,*
3. *an A_∞ structure*

$$m_k : CW^\bullet(L_0,L_1) \otimes \cdots \otimes CW^\bullet(L_{k-1},L_k) \to CW^\bullet(L_0,L_k)$$
$$m_k(a_1,\ldots,a_k) = (-1)^{\square_k} \boldsymbol{F}_{0;k,1}(a_1,\ldots,a_k);$$
$$\square_k = \sum_1^k i \cdot \deg a_i$$

Recall that $\boldsymbol{F}_{0;k,1}(a_1,\ldots,a_k)$ is given by a counting of a zero-dimensional component of a moduli space of pseudo-holomorphic discs

$$\mathcal{M}_{0;k,1}(a_0;,a_1,\ldots,a_k).$$

The proof of A_∞ relation follows from the degeneration patterns of pseudo-holomorphic discs, which corresponds to a codimension 1 boundary strata of

Gromov bordification $\overline{\mathscr{M}}_{0;k,1}(a_0;, a_1, \dots, a_k)$. In particular, we have $m_1^2 = 0$. The cohomology of a complex, denoted by

$$HW^\bullet(L_0, L_1) = H^\bullet\left(CW^\bullet(L_0, L_1; H), m_1\right)$$

, called wrapped Floer cohomology between L_0 and L_1. It does not depends on the choices of hamiltonian H or its perturbation F.

2.5 Symplectic cohomology and closed-open map

Definition 2.5.1. *A symplectic cochain complex is a $\mathbb{Z}$-graded cochain complex*

$$CH^\bullet(M; H_{S^1}) = \bigoplus_{\gamma \in \mathscr{O}(M; H_{S^1})} o_\gamma$$

graded by the degree $\deg o_\gamma$. *We will use the notation γ instead of o_γ for generators if it cause no confusion. A differential of this complex is*

$$\mathrm{d}_{CH}(o_{\gamma_1}) = (-1)^{\deg o_{\gamma_1}} \boldsymbol{F}_{1,1;0}(\gamma_1).$$

Recall that $\boldsymbol{F}_{1,1;0}(\gamma_1)$ is given by a counting of a zero-dimensional component of a moduli space of pseudo - holomorphic annulus

$$\mathscr{M}_{1,1;0}(\gamma_1, \gamma_0).$$

The proof of $\mathrm{d}^2 = 0$ follows from the degeneration patterns of pseudo-holomorphic annulus, which is a codimension 1 boundary strata of $\overline{\mathscr{M}}_{1,1;0}(\gamma_1, \gamma_0)$. In particular, we have $m_1^2 = 0$. The cohomology of a complex, denoted by

$$SH^\bullet(M) = H^\bullet\left(CH^\bullet(L_0, L_1; H), \mathrm{d}_{CH}\right)$$

is called symplectic cohomology of M. It is an invariant of M and does not depend on the specific choices of hamiltonians or its perturbation.

A chain level operation

$$CH^\bullet(M)^{\otimes 2} \to CH^\bullet(M) \tag{2.5.1}$$

$$(\gamma_1, \gamma_2) \mapsto (-1)^{\deg \gamma_1} \mathbf{F}_{2,1;0}(\gamma_1, \gamma_2) \tag{2.5.2}$$

induces a ring structure on its cohomology.

Symplectic cohomology ring acts on the wrapped Fukaya category, just like a general ring acts on its modules. Let's start with the definition of Hochschild cohomology of A_∞ category.

Definition 2.5.2. *A closed-open map is a map*

$$\mathrm{CO} : CH^\bullet(M) \to CC^\bullet(\mathscr{W}\mathscr{F}(M), \mathscr{W}\mathscr{F}(M)) \tag{2.5.3}$$

$$CO(\gamma)(a_1, \ldots, a_n) := (-1)^{\square_k} \boldsymbol{F}_{1;n,1}(\gamma; a_1, \ldots, a_n, a_0) \tag{2.5.4}$$

$$\square_k = \sum i \cdot \deg a_k \tag{2.5.5}$$

A degeneration pattern of a moduli space $\overline{\mathscr{M}}_{1;n,1}(\gamma; a_1, \ldots, a_n, a_0)$ proves that CO is a cochain map.

Chapter 3

New A_∞ category $\mathscr{C}_\Gamma$

For a chosen symplectic cohomology class $\Gamma \in SH^0(M)$, we construct a new A_∞ category $\mathscr{C}_\Gamma$ on which the action of Γ vanishes.

3.1 Popsicles with interior markings

Abouzaid-Seidel introduced the notion of a popsicle and popsicle maps to define a homotopy direct limit version of wrapped Fukaya category [AS10]. A popsicle is a punctured disc with interior marked points which can move along special lines on the disc. This interior marked points (called sprinkles) were not used as inputs in [AS10], but as a tracking device to to write various continuation maps (to increase slopes according to weights) in a consistent way. In particular, these marked points were allowed to coincide. Therefore a popsicle with one sprinkle provides continuation maps, which are expected to be isomorphisms in Floer cohomology. Abouzaid-Seidel has described the compactification of moduli of popsicles and the signs for associated A_∞-operations.

We will use a variation of the notion of popsicle, but our usage is completely different from [AS10]. We will use interior marked point as places for actual inputs (given by a symplectic cohomology class). Therefore, the compactification of the moduli space of popsicles is somewhat different from [AS10] in that if interior marked point collide, we introduce sphere bubbles as in the standard Floer theory. Also, we do not use any weights. For example, a popsicle with one sprinkle will be regarded as a quantum cap action of a symplectic cohomology class and the images of quantum cap action will vanish in the new cohomology theory.

A relevant moduli space consists of popsicles.

Definition 3.1.1 (See [AS10])**.** *A ϕ-flavoured popsicle with interior markings is a disc D^2 with following decorations;*

1. ***boundary markings**: its boundary carries a single outgoing marking denoted by z_0 and n incomming markings denoted by $z_1, \dots, z_n$.*
2. ***popsicle sticks**: geodesic l_i connecting each $z_i (i \geq 1)$ and z_0*
3. ***flavour**: a finite index set F and a set map*

$$\phi : F \to \{1, \dots, n\}.$$

4. ***sprinkles**: a function*

$$x : F \to l_{\phi(f)}$$

such that if $\phi(f_1) = \phi(f_2)$, then $x(f_1) \neq x(f_2)$ for $f_i \in F$.

We called it stable if $n + |F| \geq 2$. We denotes a moduli space of ϕ-flavoured popsicles with n boundary incoming marked points modulo automorphism by $P_{n,F,\phi}$. Also, we denote

$$Sym^\phi \subset S_F$$

a subgroup of a symmetry group S_F which stabilizes ϕ.

Geometrically, a flavour map ϕ and sprinkle x are nothing but an assignment of an interior marked point $x(f)$ on a geodesic $l_{\phi(f)}$. Since we have no conditions on ϕ, two or more interior markings are allowed to be on a same popsicle stick. But because of the additional restriction on x, all interior markings are different from each other. See Figure 3.1. We list some of the basic properties of this moduli space. Since $x(f)$ are points on a infinite geodesic, we can identify them with real numbers. Consider a fiber of the forgetful map

$$\pi : P_{n,F,\phi} \to S_{0;n,1}.$$

When $n \geq 2$, it is a open subset

$$\left\{\vec{x} \in \mathbb{R}^{|F|} \mid x(f_1) \neq x(f_2) \text{ when } \phi(f_1) = \phi(f_2)\right\}.$$

If we reverse this maps, we get an embedding

$$(\pi, x) : P_{n,F,\phi} \to S_{0;n,1} \times \mathbb{R}^{|F|}.$$

When $n = 1$, then there is only one popsicle stick and a translation of a holomorphic strip becomes an automorphism of a popsicles. Therefore the fiber of a forgetful map is

$$\left\{\vec{x} \in \mathbb{R}^{|F|} \mid x(f_1) \neq x(f_2)\right\} / \mathbb{R}.$$

.t which can be viewed as a subspace of $\mathbb{R}^{|F|-1}$. Notice that Sym^ϕ is trivial if and only if $|F|$ is injective. If not, any transposition of F is an orientation reversing automorphism of $P_{n,F,\phi}$

3.2 Compactification

We keep following [AS10] where the compactification and transversality argument has been established for holomorphic popsicles. But as we remark in the last section, the Gromov bordification $\overline{P}_{n,F,\phi}$ is larger then the original reference. Its boundary strata contains sphere bubbles as depicted in Figure 3.1. Although we won't use that extra component, we include a brief description of it for sake of completeness.

Definition 3.2.1. *A rooted ribbon tree is a tree T with*

- *a root and leaves: $d+1$ semi-infinite edges with a preferred choice of one among them. The preferred one is called the root, and the rest is called leaves.*
- *ribbon structure: a cyclic order on adjacent edges for each vertex v of T.*

The root and leaves determine a direction on edges. Each vertex v has a single adjacent edge e_0 eminating from the root, and the rest are cyclicly ordered as $\{e_1, \ldots, e_{val(v)-1}\}$.

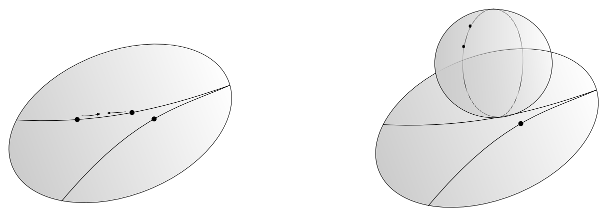


Figure 3.1: (left) Example of $P_{2,F,\phi}$ with $|F| = 3$. (right) A sphere bubble occurs when two or more sprinkles on a same popsicle stick collide

At first, let us describe a model for sphere bubbles.

Definition 3.2.2. *Let T be a rooted tree with no leaves. An F-flavoured icecream modelled on T consists of spheres $\mathbb{P}^1_w$ for each vertex w with the following decorations.*

- *an anti-holomorphic involution $\tau_w : \mathbb{P}^1_w \to \mathbb{P}^1_w$*
- *$val(w)$-special points which is invariant under τ_w and respects a cyclic order at w.*
- *decomposition of a set of flavour $F = \bigsqcup_w F_w$;*
- *a sprinkle function $x_w : F_w \to \mathbb{P}^1_w$ whose image is also τ_w-invariant and disjoint from special points.*

We call F-flavoured icecream is stable if there are more than three special points on each $\mathbb{P}^1_w$.

The reader would immediately notice that a ϕ-flavoured icecream is just a model for a sphere bubble when two or more sprinkles on a same popsicle stick collide. A tree T only determines a configuration of a sphere bubble so it has no leaves. Extra markings other than nodal points are determined from a sprinkle function x_w. An involution τ comes form the following reason; a popsicle stick can be considered as a fixed locus of an anti-holomorphic involution on a disc. Whenever several sprinkles collide, a sphere bubble also carries an involution τ. All nodal points and sprinkles should be τ-invariant.

Definition 3.2.3. *A ϕ-flavoured broken popsicle with icecream on it modeled on a rooted tree T consists of*

- ***decomposition of*** F*: decomposition*
$$F = \bigcup_v F_v$$
and
$$F_v = F'_v \sqcup \bigsqcup_i F'_{v,i}$$
- ***decomposition of*** ϕ*: a map*
$$\phi_v := \phi|_{F_v} : F_v \to \{1, \ldots, val(v) - 1\}$$
satisfying the following two conditions.
  1. *for each $f \in F'_v$, the vertex v must lie on the unique path from the root to $e_{\phi_v(f)}$ at v;*
  2. *an image $\phi_v(F'_{v,i})$ is a single point.*
- ***popsicles****: an assignment of ϕ_v-flavoured popsicle on each v such that the sprinkle map x_v is injective on F'_v and constant on $x_v(F'_{v,i})$. Images $x_v(F'_v)$ and $x_v(F'_{v,i})$ are different.*
- ***icecream****: a stable $F'_{v,i}$-flavoured icecream structure modeled on some rooted tree $T'_{v,i}$ with no leaves for each (v,i).*

A ϕ-flavoured broken popsicle with icecream on it is called stable if all popsicles and icecreams are stable.

Although the definition looks complicated, the geometric intuition should be clear. A decomposition of F_v consists of two parts; F'_v is a part on which ϕ is injective, and we assign an ordinary popsicle structures according to its image. On the other hand, $F'_{v,i}$ is a set of sprinkles that collides at the point $\phi(F'_{v,i})$. We attach a sphere bubble, or icecream on that point. Notice that $|F'_{v,i}| \geq 2$ as soon as it is stable.

A moduli space of ϕ-flavoured broken popsicle with icecream modeled on T is a product

$$P^T_{n,F,\phi} = \prod_v P_{val(v)-1,F_v,\phi_v} \times \prod_{T'_{v,i}} \mathbb{R}^{|F'_{v,i}|+|edge(T'_{v,i})|-3|vert(T'_{v,i})|}$$

Take the disjoint union of those spaces, and denote it by

$$\overline{P}_{n,F,\phi} := \bigsqcup_{T,F=\bigsqcup F_v} P^T_{n,F,\phi}.$$

Proposition 3.2.4. *$\overline{P}_{n,F,\phi}$ is a compact smooth manifold with corners.*

Proof. The boundary strata is a mixture of two disjoint degenerations; one is when an underlying disc component breaks into several pieces, and the other is when several sprinkles collide.

The first part can be covered by the result of [AS10]. If we forget about icecream structure and simply allow a sprinkle function x_v may not be injective, then the corresponding moduli is the same as their moduli spaces of ϕ-flavoured popsicles. They construct an algebro-geometric model (called holomorphic lollipops) for such moduli spaces and prove that a standard gluing procedure along strip-like end gives a structure of a smooth manifold with corner on the moduli space.

Then, a second kind of degeneration can be covered easily. This is essentially a compactification of a configuration space of points on S^1. (See Fulton-Macpherson [FM94]). Consider a fiber of a forgetful map

$$\pi_v : P_{val(v)-1,F_v,\phi_v} \to S_{0;val(v)-1,1}$$

It is an open complement of $\mathbb{R}^{|F_v|}$ given by

$$\left\{\vec{x} \in \mathbb{R}^{|F_v|} : x_v(f_1) \neq x_v(f_2), \ \ \forall f_i \in F'_{v,i}\right\}$$

A value $x_v(F'_{v,i})$ determines a limit point on a naive compactified fiber $\mathbb{R}^{|F|}$. We perform a consecutive oriented real blow-up on the locus where two or more coordinate coincides until all coordinates are finally distinguished to each other. A rooted tree $T'_{v,i}$ corresponds exactly to a possible boundary strata of this blowups. The number of vertices of $T'_{v,i}$ determines the number of blow-ups you perform to reach that strata. A value of sprinkles x_w then determines a coordinates of a moduli.

A real oriented blow-ups of a smooth compact manifold with corners is again a smooth compact manifold with corners. We finish the proof. $\square$

The structure of a manifold with corners are compatible to a canonical inclusion

$$\mathscr{P}_{n,F,\phi} \subset \overline{S}_{|F|;n,1}.$$

We leave it to an interested reader.

Definition 3.2.5. *Let $a_i \in CF^\bullet(L_{i-1}, L_i)$ and $\Gamma \in SH^0(M)$. Define*

$$\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$$

be a compactified moduli space of pseudo-holomorphic maps

$$\left\{u : S \to M : S \in \overline{P}_{n,F,\phi}\right\}$$

satisfies

- *a boundary segment from z_{i-1} to z_i goes to L_i,*
- *a boundary marking z_i goes to a_i,*
- *all interior markings are asymptotic to Γ.*

It can be described as a submanifold with corners

$$\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n) \subset \overline{\mathscr{M}}_{|F|;n,1}(\Gamma, \ldots, \Gamma; a_1, \ldots, a_n, a_0)$$

cut out by a popsicle conditions on interior marked points.

A standard compactness and transversality argument now can be applied. Notice that we can choose a Floer data consistently for a family of domains. The moduli space is still a manifold with corners, so we may extend it inductively form the lowest dimensional strata.

Lemma 3.2.6. *For a generic choice of universal and consistent Floer data,*

1. *The moduli spaces* $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ *are smooth and compact.*
2. *For a given input* Γ *and* a_i, $i = 1, \ldots, n$, *there are only finitely many* a_0 *for which* $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ *is non-empty .*
3. *It is a manifold of dimension*

$$|F| + n - 2 + \deg a_0 - \sum_{i=1}^{n} \deg a_i$$

Proof. A compactness and transversality argument is mostly the same as before. A standard index formula tells us that

$$\begin{aligned}
\dim_{\mathbb{R}} \overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n) &= \dim_{\mathbb{R}} \overline{\mathscr{M}}_{|F|;n,1}(\Gamma, \ldots, \Gamma; a_1, \ldots, a_n, a_0) - |F| && (3.2.1)\\
&= (2|F| + n - 2) + \deg a_0 - \sum_{i=1}^{n} \deg a_i - |F| \cdot \deg \Gamma && (3.2.2)\\
&= |F| + n - 2 + \deg a_0 - \sum_{i=1}^{n} \deg a_i. && (3.2.3)
\end{aligned}$$

□

We denote an orientation operator associated to the zero-dimensional component of $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ by

$$m^\Gamma_{n,F,\phi}.$$

In particular, $m^\Gamma_{n,F} = m_n$ if F is an empty set. A degree of this operator is $2 - n - |F|$

3.2.1 A_∞ category $\mathscr{C}_\Gamma$

In this section, we construct a new A_∞ category $\mathscr{C}_\Gamma$. We start with the following important observation.

Proposition 3.2.7. *If* $\phi : F \to \{1, \ldots, n\}$ *is not injective, then* $m^\Gamma_{n,F,\phi}$ *vanishes.*

Proof. It means that at least one popsicle stick carries more than two interior markings, which also means ϕ is not injective. Then Sym^ϕ contains a nontrivial transposition. Since we put a same class Γ for all interior markings, the transposition extends to $\overline{\mathscr{P}}_{n,F,\phi}(\Gamma; a_1, ..., a_n)$ also. It induces an orientation-reversal automorphism on $\overline{\mathscr{P}}_{n,F,\phi}$. Therefore the contribution of this moduli space should vanish. □

Now we can focus on the case when $\phi : F \to \{1,\dots,n\}$ is injective. Then F can be considered as a subset $\{i_1,\dots,i_k\} \subset \{1,\dots,n\}$. In this case, we omit a notation ϕ and simply write $\overline{\mathscr{P}}_{n,F}$ and $m^\Gamma_{n,F}$.

Definition 3.2.8. *An admissible cut of F consists of*

1. $n_1, n_2 \geq 1$ *such that* $n_1 + n_2 = n+1$
2. *a number* $i \in \{1,\dots,n\}$
3. $F_1 \subset \{1,\dots,n_1\}$ *and* $F_2 \subset \{1,\dots,n_2\}$ *such that* $|F_1| + |F_2| = |F|$

satisfies the following property;

- $F \supset \{k | k \in F_1, k < i\}$ *and* $F \supset \{k + n_2 - 1 | k \in F_1, k > i\}$
- $F \supset \{k + i - 1 | k \in F_2\}$

If $i \notin F_1$*, then this completely recovers* F*. Otherwise,* F *has one more element among* $\{i, i+1, \dots, (i+n_2-1)\}$.

An admissible cut describes a stratum $P_{n_1,F_1} \times P_{n_2,F_2}$ of a moduli space $\overline{P}_{n,F}$. They describes precisely codimension 1 strata whose associated sprinkle $\phi_j : F_j \to \{1,\dots,n_j\}$ $(j = 1,2)$ is still injective. Combined with 3.2.7, we get a quadratic relation

$$\sum_{\forall \text{admissible cuts}} (-1)^{\square} m^\Gamma_{n_1,F_1}(a_1,\dots,a_{i-1}, m_{n_2,F_2}(a_i,\dots,a_{i+|F_2|}), a_{i+1+|F_2|},\dots,a_{n_1+n_2-1})$$
$$= 0$$

Now we are ready to define a new A_∞ category.

Definition 3.2.9. *Let* $\Gamma \in SH^0(M)$*. A category* $\mathscr{C}_\Gamma$ *consists of*

1. *a set of objects* $Ob(\mathscr{C}_\Gamma) = Ob(\mathscr{W}\mathscr{F}(M))$.

2. *morphisms between two objects*

$$\mathrm{Hom}_{\mathscr{C}_\Gamma}(L_1, L_2) = CW(L_1, L_2) \oplus CW(L_1, L_2)[1]$$

We denotes the element of this complex by $c := a + \epsilon b$ *with* $deg\epsilon = -1$

3. *An* A_∞ *structure* $\{M_n\}_{n=1}^{\infty}$ *is given as follows. We may write*

$$M_n(c_1, \cdots, c_n) = M_n^a(c_1, \cdots, c_n) + \epsilon M_n^b(c_1, \cdots, c_n)$$

   (a) *Suppose* $c_i = a_i$ *for all* i *(all the inputs do not have* ϵ *components), then we set*

$$M_n(a_1, \cdots, a_n) = m_n(a_1, \cdots, a_n)$$

   where $\{m_n\}$ *is the* A_∞*-operation for* $\mathscr{W}\mathscr{F}(M)$.

   (b) *Suppose* $c_i = \epsilon b_i$ *for* $i \in \{i_1, \ldots, i_k\}$*, and* $c_i = a_i$ *for* $i \notin \{i_1, \ldots, i_k\}$*. Then we set*

$$F = \{i_1, \ldots i_k\}, \quad \widehat{F}^j = \{i_1, \ldots, \widehat{i_j}, \ldots, i_k\},$$

   and define

$$M_n(c_1, \ldots, c_n) = M_n^a(c_1, \ldots, c_n) + \epsilon M_n^b(c_1, \ldots, c_n)$$

$$M_n^a(c_1, \ldots, c_n) = (-1)^{\epsilon_a} m_{n,F}^{\Gamma}(a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, b_{i_k}, \ldots, a_n)$$

$$M_n^b(c_1, \ldots, c_n) = \sum_{j=1}^{k} (-1)^{\epsilon_j + \epsilon_{b,j}} m_{n,\widehat{F}^j}^{\Gamma}(a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, b_{i_k}, \ldots, a_n)$$

   If we use the notion x_i *to denote* b_i *for* $i \in F$ *and* a_i *for* $i \notin F$*. Denote by* $|x|' = |x| - 1$*. Then*

$$\begin{aligned}
\epsilon_a &= \sum_j |x_j| + \sum_{f \in F, l > f} (|x_l| - 1) \\
\epsilon_{b,j} &= \sum_j |x_j| + \sum_{f \in \widehat{F}^j, l > j} |x_l|' \\
\epsilon_j &= \sum_{l=1}^{j-1} |x_l|'
\end{aligned}$$

Remark 3.2.10. *Although geometric setting of this A_∞-category and that of Abouzaid-Seidel [AS10] (homotopy limit chain complex of wrapped Fukaya category) are completely different, the resulting algebraic structures has strong similarities because both are based on some versions of "popsicle" moduli spaces. In particular, we can use the sign analysis of popsicle moduli space of [AS10] to have the same sign as above.*

As a sanity check, we check the degree of A_∞-operations. Recall the degree of $m^\Gamma_{n,F}$ is $2-n-|F|$. Additional degree shift $-|F|$ comes from interior markings. We correspondingly shift our inputs b_i for each interior markings by multiplying ϵ. Therefore, a degree of M_n becomes $2-n$.

Proposition 3.2.11. *$\mathscr{C}_\Gamma$ is an A_∞ category. Namely, for any composable $(c_1,\cdots,c_n)$, we have*

$$\sum_{n_1+n_2=n+1}(-1)^{\sum_{j=1}^{i-1}|c_j|'}M_{n_1}(c_1,\cdots,c_{i-1},M_{n_2}(c_i,\cdots,c_{i+n_2-1}),\cdots,c_n)=0$$

Proof. We check the identity on each component of the output. We first show that

$$\sum\left(M^a_{n_1}(\cdots,M^a_{n_2}(\cdots),\cdots)+M^a_{n_1}(\cdots,M^b_{n_2}(\cdots),\cdots)\right)=0.$$

This identify follows from the compactification of popsicle moduli space. Namely, A codimension one statra of popsicle moduli space corresponds to a term in the above equation. In Figure 3.2, we illustrated corresponding broken popsicles in the same order for the case $|F|=4$.

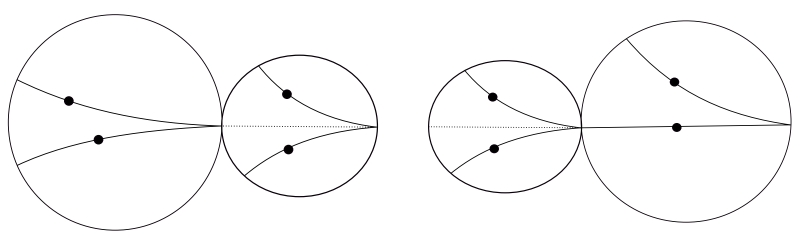


Figure 3.2: A_∞-identity with a-output

Next we show that

$$\sum\left(M^b_{n_1}(\cdots,M^a_{n_2}(\cdots),\cdots)+M^b_{n_1}(\cdots,M^b_{n_2}(\cdots),\cdots)\right)=0.$$

This identify follows from the compactification of popsicle moduli space for $\widehat{F}^j$ for all j. In Figure 3.3, we illustrated corresponding broken popsicles in the same order for the case $|F| = 4$ and $j = 1$.

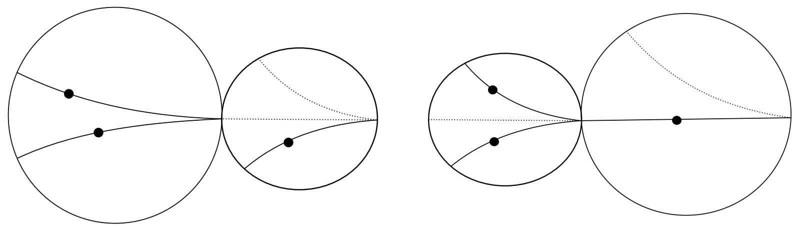


Figure 3.3: A_∞-identity with b-output

Now let us explain the signs. We will be very brief, since how to construct orientation of Fukaya category is by now well-understood. Also, Abouzaid-Seidel already carried out detailed sign analysis for popsicles and popsicle maps, which we can easily adapt to our setting. In particular note that the signs appearing in the definition of A_∞-operation here are the same as Section 3h [AS10].

First, recall that we are using the sprinkle as a place for interior Γ-insertion whereas in [AS10] sprinkles are just a marker for some other data. Hence orientation for the latter for a sprinkle is given by the orientation of $\mathbb{R}$ (the popsicle stick), but in our case, we need $\mathbb{R} \otimes o_\Gamma$ where o_Γ is the orientation operator of the Reeb orbit Γ. It is important that our symplectic cohomology insertion Γ has even degree so that it does not affect any sign for switching places. We refer readers to [AS10] for detailed explanation for signs, and leave the adaptation as an exercise. $\square$

3.3 Cohomology category

In this subsection, we describe $\mathscr{C}_\Gamma$ at its cohomology level. A differential is given by

$$M_1(a+\epsilon b) = \left(m_1(a) + m^\Gamma_{1,\{1\}}(b)\right) + \epsilon m_1(b), \quad a, b \in CW(L_1, L_2).$$

Proposition 3.3.1. *Up to homotopy, we have*

$$m^\Gamma_{1,1}(b) = m_2\left(b, \mathrm{CO}_{L_2}(\Gamma)\right)$$

where $CO_{L_2} : SH^\bullet(M) \to CW^\bullet(L_2, L_2)$. *is a word-length zero component of the closed-open map.*

Proof. Consider the moduli space of popsicles $P_{1,\{1\}}$. The moduli space is isolated. It is a moduli space of disc with one outgoing boundary marking $z_0 = 1$, one incoming boundary marking $z_1 = -1$ and also a single interior marking $x_0 = 0$. Now consider a 1-parameter family of moduli space of holomorphic discs with

- one outgoing boundary marking z_0 at 1
- one incoming boundary marking z_1 at -1
- one moving interior marking x_0 at $-it, \quad t \in [0,1]$

At $t = 0$, we get $P_{1,\{1\}}$. At $t = 1$, we get a moduli space of discs with disc bubbles containing interior marked points. See Figure 3.4. It corresponds to a disc moduli space governing $m_2\left(b, \mathrm{CO}_{L_2}(\Gamma)\right)$. $\square$

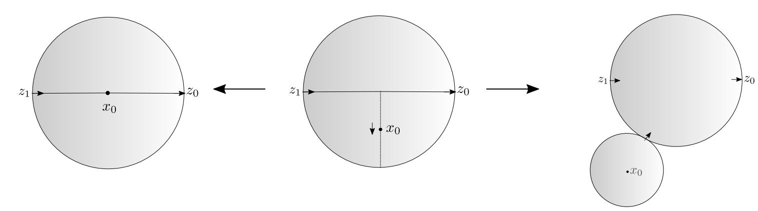


Figure 3.4: Homotopy between $m^{\Gamma}_{1,\{1\}}$ and $\mathrm{CO}(\Gamma)$

Corollary 3.3.2. *As a complex,*

$$\mathrm{Hom}^\bullet_{\mathscr{C}_\Gamma}(L_1, L_2) \simeq \mathrm{Cone}\left(CW^\bullet(L_1, L_2) \xrightarrow{m_2(-.\mathrm{CO}_{L_2}(\Gamma))} CW^\bullet(L_1, L_2) \right)$$

Therefore a category $\mathscr{C}_\Gamma$ is an A_∞ category on which an action of Γ by a closed-open map vanishes homotopically. This observation becomes even more clear if we formulate it in the language of bimodules. Let's restrict our input to be of the form

$$(a_1 \otimes \cdots \otimes a_i \otimes \underline{b} \otimes a_{i+1} \otimes \cdots \otimes a_n) \in (\mathscr{W}\mathscr{F}^{\otimes i}) \otimes \underline{\mathscr{W}\mathscr{F}} \otimes (\mathscr{W}\mathscr{F}^{\otimes n-i})$$

Here we underline the middle component to emphasize that we consider ϵb as an element of bimodules.

Definition 3.3.3 (Quantum cap action of Γ)**.** *A cocain level Quantum cap action of* Γ *is an* A_∞ *pre-bimodule map defined by*

$$\cap\Gamma : T(\mathscr{W}\mathscr{F}) \otimes \underline{\mathscr{W}\mathscr{F}} \otimes T(\mathscr{W}\mathscr{F}) \to \mathscr{W}\mathscr{F} \tag{3.3.1}$$

$$(a_1, \ldots, a_i, \underline{b}, a_{i+1}, \ldots, a_n) \mapsto M^a_{k+1}(a_1, \ldots, a_i, \epsilon b, a_{i+1}, \ldots, a_n). \tag{3.3.2}$$

Indeed, it is a bimodule homomorphism from $\mathscr{W}\mathscr{F}(M)$ to itself. We only have to show that the differential of this pre-bimodule map vanishes, which is just a part of **??** when $|F| = 1$. Therefore we found a distinguished triangle

$$\mathscr{W}\mathscr{F}(M) \xrightarrow{\cap\Gamma} \mathscr{W}\mathscr{F}(M) \longrightarrow \mathscr{C}_\Gamma \longrightarrow$$

of bimodules.

Remark 3.3.4. *A cochain level* $\cap\Gamma$ *descend to a cohomology category* $H(\mathscr{W}\mathscr{F}(M))$*, which is the standard quantum cap action as explained in [Aur07].*

On the other hand, a complex of bimodule homomorphism $\mathrm{Hom}_{\mathscr{A}-\mathscr{A}}(\Delta_{\mathscr{A}}, \Delta_{\mathscr{A}})$ *is one of the presentation of Hochschild cohomology* $HH^\bullet(\mathscr{A}, \mathscr{A})$*. In [Gan13], a bimodule version of closed-open map, called two-pointed open-closed map,*

$${}^2\mathrm{CO} : SH^\bullet(M) \to {}^2CC^\bullet(\mathscr{W}\mathscr{F}(M), \mathscr{W}\mathscr{F}(M))$$

was constructed. One can check directly that the quantum cap action $\cap\Gamma$ *coincides to* ${}^2\mathrm{CO}(\Gamma)$

Intuitively,

- objects of $\mathscr{C}_\Gamma$ are a twisted complex

$$\left(L \xrightarrow{\mathrm{CO}(\Gamma)} L \right), \quad L \in \mathscr{W}\mathscr{F}(M).$$

In many cases, they can be realized as a geometric surgery of L with itself along CO(Γ).

- the space of morphisms is a "half" of the original one. It consists of

$$"a+\epsilon b" = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in CW^\bullet \left(\; L_1 \xrightarrow{\mathrm{CO}(\Gamma)} L_1 \;\; , \;\; L_2 \xrightarrow{\mathrm{CO}(\Gamma)} L_2 \; \right)$$

Of course, this intuitive analogy is simply not true in all possible ways. Let's use more than one interior marking. Unlike $\widetilde{m}_1$, $\widetilde{m}_2$ operation is already counter-intuitive.

$$m_2(\epsilon b_1, \epsilon b_2) = m^\Gamma_{2,\{1,2\}}(b_1, b_2) + \epsilon(\text{extra term}).$$

One might expect $\widetilde{m}_2(\epsilon b_1, \epsilon b_2)$ vanishes, according to the intuition. It is not true. In fact, we will exibit an example that $\widetilde{m}_2(\epsilon b_1, \epsilon b_2) = 1$ at the level of cohomology.

3.4 Example: M_2-operation

Let us examine the Leibniz rule for the input $(a, \epsilon b)$. Namely, we want to verify

$$M_1(M_2(a, \epsilon b)) + M_2(M_1(a), \epsilon b) + (-1)^{|a|'} M_2(a, M_1(\epsilon b)) = 0. \tag{3.4.1}$$

From the definition

$$M_2(a, \epsilon b) = m^\Gamma_{2,\{2\}}(a, b) + \epsilon m_2(a, b)$$

$$M_1(\epsilon b) = m^\Gamma_{1,\{1\}}(b) + \epsilon m_1(b)$$

We have

$$\begin{aligned}
M_1(M_2(a,\epsilon b)) &= M_1(m^\Gamma_{2,\{2\}}(a,b) + \epsilon m_2(a,b)) \\
&\quad = m_1(m^\Gamma_{2,\{2\}}(a,b)) + \left(m^\Gamma_{1,\{1\}}(m_2(a,b)) + \epsilon m_1(m_2(a,b))\right) \\
M_2(M_1(a),\epsilon b) &= M_2(m_1(a),\epsilon b) = m^\Gamma_{2,\{2\}}(m_1(a), b) + \epsilon m_2(m_1(a), b) \\
M_2(a, M_1(\epsilon b)) &= M_2(a, m^\Gamma_{1,\{1\}}(b) + \epsilon m_1(b)) \\
&= m_2(a, m^\Gamma_{1,\{1\}}(b)) + m^\Gamma_{2,\{2\}}(a, m_1(b)) + \epsilon M_2(a, m_1(b))
\end{aligned}$$

If we collect the terms with ϵ in (3.4.1), we obtain the original A_∞-identity

$$\epsilon\big(m_1(m_2(a,b)) + m_2(m_1(a),b) + (-1)^{|a|'} m_2(a, m_1(b))\big) = 0.$$

Collecting the terms without ϵ in (3.4.1), we get the following

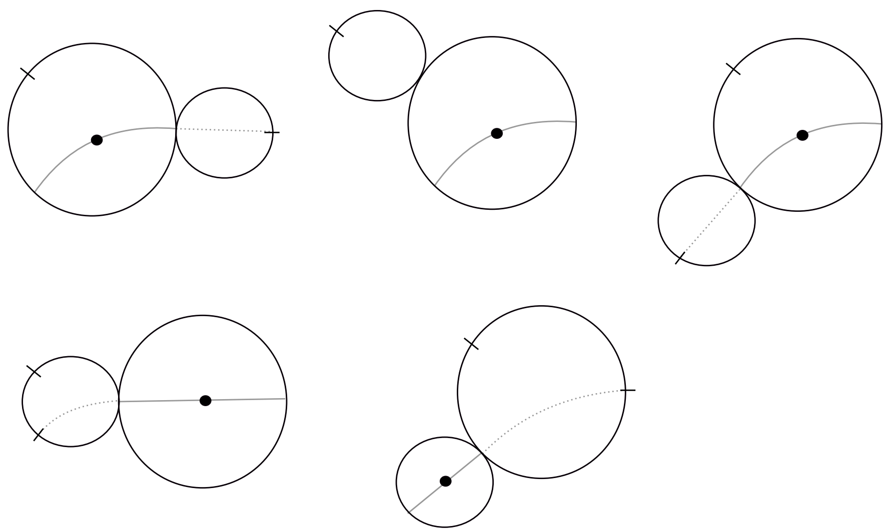


Figure 3.5: Leibniz rule for the inputs $(a, \epsilon b)$

$$m_1 m^\Gamma_{2,\{2\}}(a,b) + m^\Gamma_{2,\{2\}}(m_1(a),b) + m^\Gamma_{2,\{2\}}(a, m_1(b))$$

$$+ m^\Gamma_{1,\{1\}}(m_2(a,b)) + m_2(a, m^\Gamma_{1,\{1\}}(b)) = 0.$$

These terms correspond to the codimension one degenerations (given by disc bubblings) in Figure 3.5. Here dotted lines just indicate paths to the 0-th vertex, and do not give any restriction to the domain. Hence one may remove dotted lines to find the corresponding A_∞-operations.

Let us examine Leibniz rule for the input $(\epsilon b_1, \epsilon b_2)$. Namely, we want to verify

$$M_1(M_2(\epsilon b_1, \epsilon b_2)) + M_2(M_1(\epsilon b_1), \epsilon b_2) + (-1)^{|b_1|} M_2(\epsilon b_1, M_1(\epsilon b_2)) = 0. \quad (3.4.2)$$

We have

$$\begin{aligned}
M_1(M_2(\epsilon b_1,\epsilon b_2)) &= M_1(m^\Gamma_{2,\{1,2\}}(b_1,b_2)+\epsilon m^\Gamma_{2,\{2\}}(b_1,b_2)+\epsilon m^\Gamma_{2,\{1\}}(b_1,b_2)) \\
&= m_1(m^\Gamma_{2,\{1,2\}}(b_1,b_2))+m^\Gamma_{1,\{1\}}(m^\Gamma_{2,\{2\}}(b_1,b_2)+m^\Gamma_{2,\{1\}}(b_1,b_2)) \\
&\quad +\epsilon m_1\big(m^\Gamma_{2,\{2\}}(b_1,b_2)+m^\Gamma_{2,\{1\}}(b_1,b_2)\big) \\
M_2(M_1(\epsilon b_1),\epsilon b_2) &= M_2(m^\Gamma_{1,\{1\}}(b_1)+\epsilon m_1(b_1),\epsilon b_2) \\
&= m^\Gamma_{2,\{2\}}(m^\Gamma_{1,\{1\}}(b_1),b_2)+\epsilon m_2(m^\Gamma_{1,\{1\}}(b_1),b_2) \\
&\quad +m^\Gamma_{2,\{1,2\}}(m_1(b_1),b_2)+\epsilon m^\Gamma_{2,\{2\}}(m_1(b_1),b_2)+\epsilon m^\Gamma_{2,\{1\}}(m_1(b_1),b_2) \\
M_2(\epsilon b_1,M_1(\epsilon b_2)) &= M_2(\epsilon b_1,m^\Gamma_{1,\{1\}}(b_2)+\epsilon m_1(b_2)) \\
&= m^\Gamma_{2,\{1\}}(b_1,m^\Gamma_{1,\{1\}}(b_2))+\epsilon m_2(b_1,m^\Gamma_{1,\{1\}}(b_2)) \\
&\quad +m^\Gamma_{2,\{1,2\}}(b_1,m_1(b_2))+\epsilon m^\Gamma_{2,\{2\}}(b_1,m_1(b_2))+\epsilon m^\Gamma_{2,\{1\}}(b_1,m_1(b_2))
\end{aligned}$$

The following figure 3.6 describes the terms without ϵ in the above (in the same order). It is not hard to see that these arise from codimension boundary

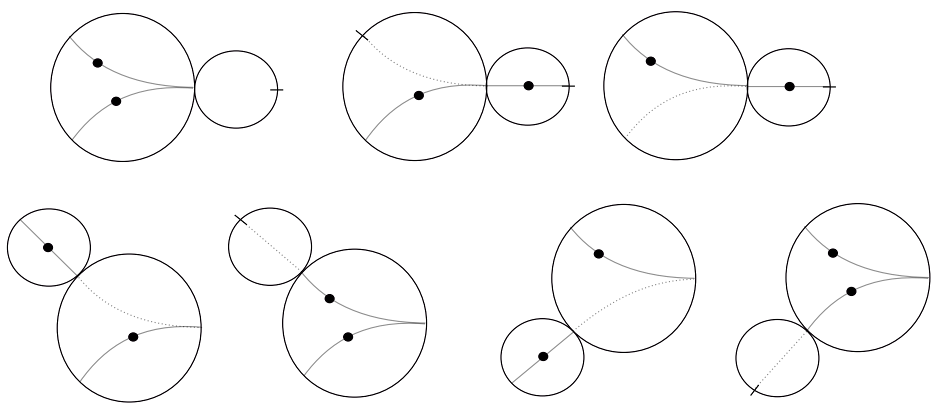


Figure 3.6: Leibniz rule for the inputs $(\epsilon b_1,\epsilon b_2)$

strata of $\overline{P}_{2,\{1,2\}}$. This The terms with ϵ are similar. As we can see from this example, there is no reason to expect $M_1\circ M_2(\epsilon a,\epsilon b)=0$. This make $\{M_k\}$ a bit counter-intuitive.

Chapter 4

Algebro-geometric counterpart

We discuss classical algebro-geometric operation of restricting to a hypersurface in D^bCoh and MF. We will later show that monodromy of wrapped Fukaya category of a Milnor fiber is mirror to this restriction operation.

4.1 Restricting to a hypersurface in D^bCoh

Let S be an algebra. Choose an element

$$g \in Z(S) \cong HH^0(S,S)$$

The bimodule $S \xrightarrow{g} S$ is quasi-isomorphic to an ideal quotient $S/(g)$, and it carries a natural algebra structure. One can directly construct DG algebra structure on the bimodule itself:

Definition 4.1.1. *Define a DG algebra*

$$\mathscr{B} := S[\epsilon] \Big/ \begin{pmatrix} \epsilon^2 = 0 \\ d\epsilon = g \end{pmatrix}, \ \ deg\,\epsilon = -1$$

Here the differential d on S is set to be zero.

One can check that multiplication is well-defined.

Further assume that S is commutative. Consider an affine variety $X = \mathrm{Spec}(S)$ and a hypersurface $Y = V(g)$ with an inclusion $i : Y \hookrightarrow X$. We have the following elementary lemma whose proof is omitted.

Lemma 4.1.2. *We have an equivalence $\mathscr{B} \simeq i_*\mathscr{O}_Y$. Moreover, we have the following.*

1. *A sheaf $\mathscr{F}$ on a hypersurface Y corresponds to an $\mathscr{B}$-module object. It is a pair $(i_*\mathscr{F}, h_{\mathscr{F}})$ where $i_*\mathscr{F}$ is a pushfoward of $\mathscr{F}$ equipped with a homotopy $h_{\mathscr{F}}$ between the zero map and a multiplication of g. It is an action of $\epsilon \in \mathscr{B}$.*

2. *Moreover,*

$$Hom_Y(\mathscr{F}_1, \mathscr{F}_2) \simeq Hom_{\mathscr{B}}((i_*\mathscr{F}_1, h_{\mathscr{F}_1}), (i_*\mathscr{F}_2, h_{\mathscr{F}_2})).$$

For the sheaf $\mathscr{O}_Y$ on Y, its pushfoward $i_*\mathscr{O}_Y$ has a simple free resolution.

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{g} \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y \longrightarrow 0$$

An action of degree -1 element ϵ, or a homotopy h, is given as follows.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X \arrow[r, "g"] \arrow[dl, "0"'] & \mathscr{O}_X \arrow[r] \arrow[dl, "id"'] & 0 \arrow[dl, "0"'] \\
0 \arrow[r] & \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \arrow[r] & 0
\end{tikzcd}$$

We can recover a category of coherent sheaves on Y in terms of X.

Theorem 4.1.3. *(See [Pre11]) Let $Y \subset X$ as before. Then*

$$DCoh(Y) \simeq \mathscr{B} - mod(DCoh(X))$$

Proof. This is a standard application of Lurie's Barr-Beck-theorem. We present an elementary proof to illustrate the idea. Since everything is affine, It is enough to consider a structure sheaf $\mathscr{O}_Y \in DCoh(Y)$. Computation shows that the morphism complex is

$$Hom^{-1}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h)) \simeq \left\{ \begin{tikzcd} \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \arrow[dl, "a_{21}"'] \\ \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \end{tikzcd} \;\middle|\; a_{21} \text{ can be arbitrary} \right\} \tag{4.1.1}$$

$$Hom^{0}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h)) \simeq \left\{ \begin{tikzcd} \mathscr{O}_X \arrow[r, "g"] \arrow[d, "a_{11}"] & \mathscr{O}_X \arrow[d, "a_{22}"] \\ \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \end{tikzcd} \;\middle|\; a_{11} = a_{22} \right\} \tag{4.1.2}$$

$$Hom^{1}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h)) \simeq \left\{ \begin{tikzcd} \mathscr{O}_X \arrow[r, "g"] \arrow[dr, "a_{12}"] & \mathscr{O}_X \\ \mathscr{O}_X \arrow[r, "g"] & \mathscr{O}_X \end{tikzcd} \;\middle|\; h \circ a_{12} = 0 \ \text{ implies } a_{12} = 0 \right\} \tag{4.1.3}$$

Therefore, $Hom^{\bullet}_{\mathscr{B}}((i_*\mathscr{O}_Y,h),(i_*\mathscr{O}_Y,h))$ is isomorphic to

$$H^{\bullet}\left(0 \longrightarrow Hom_X(\mathscr{O}_X,\mathscr{O}_X) \xrightarrow{d=g} Hom_X(\mathscr{O}_X,\mathscr{O}_X) \longrightarrow 0 \right) \simeq Hom^{\bullet}_Y(\mathscr{O}_Y,\mathscr{O}_Y).$$

□

Intuitively, objects of $\mathscr{B}-mod(DCoh(X))$ are cones

$$\left(\mathscr{F}[1] \xrightarrow{g} \mathscr{F} \right), \ \mathscr{F} \in DCoh(X).$$

It is quasi-isomorphic to a quotient $\mathscr{F}/(g)$. And the space of morphisms is a half of the original one. It consists of

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in \mathrm{Hom}^{\bullet}_X\left(\mathscr{F}_1[1] \xrightarrow{g} \mathscr{F}_1 \ , \ \mathscr{F}_2[1] \xrightarrow{g} \mathscr{F}_2 \right).$$

Indeed, this is closed under DG operations.

4.2 Restricting to a graph hypersurface in Matrix factorizations

Consider a non-isolated singularity of the form

$$U = U_1(x_1, \ldots, x_{n-1}) + x_n \cdot U_2(x_1, \ldots, x_{n-1}).$$

We consider a graph of some polynomial g

$$\begin{aligned} \mathbb{A}^{n-1} &\to \mathbb{A}^n \\ (x_1, \ldots, x_{n-1}) &\mapsto (x_1, \ldots, x_{n-1}, g) \end{aligned}$$

and a pull-back

$$V(x_1, \ldots, x_{n-1}) = U(x_1, \ldots, x_{n-1}, g)$$

of U along this graph. We assume V is an isolated singularities. We have a relation

$$U = V + (x_n - g)U_2$$

inside We explain how to obtain a similar relation between $\mathrm{MF}(U + x_n \cdot V)$ and $\mathrm{MF}(U)$. We start by collecting functorial properties between two matrix factorization categories, which we refer to [Orl09] and [Pos11].

Let $X = \{U = 0\} \subset \mathbb{C}^n$ and $Y = \{V = 0\} \subset \mathbb{C}^{n-1}$. We view Y as a hypersurface $\{g = x_n\} \subset X$. A closed embedding $Y \hookrightarrow X$ is proper and has a finite tor-dimension. A usual adjoint pair of functors (i^*, i_*) extends to categories of singularities.

$$i^* : D^b_{sg}(X) \longleftrightarrow D^b_{sg}(Y) : i_*$$

On the other hand, there is Orlov's equivalences

$$\mathrm{MF}(U) \simeq \overline{D^b_{sg}(X)}, \;\; \mathrm{MF}(V) \simeq \overline{D^b_{sg}(Y)}$$

Here, $\overline{C}$ denotes Karoubi completion of a category C. This functor sends

$$M = (\; M^{odd} \underset{\phi_{01}}{\overset{\phi_{10}}{\rightleftarrows}} M^{even} \;) \mapsto \mathrm{coker}(\phi_{10})$$

We have an induced pair

$$i^* : \mathrm{MF}(U) \longleftrightarrow \mathrm{MF}(V) : i_*$$

Proposition 4.2.1. *Let*

$$M = \left(M^{odd} \underset{\phi_{01}}{\overset{\phi_{10}}{\rightleftarrows}} M^{even} \right) \in \mathrm{MF}(U),$$

$$N = \left(N^{odd} \underset{\psi_{01}}{\overset{\psi_{10}}{\rightleftarrows}} N^{even} \right) \in \mathrm{MF}(V)$$

Then

1. (i^*, i_*) *is an adjoint pair.*
2. $i^* M \simeq M|_{x_n=g} \in \mathrm{MF}(V)$.
3. $i_* N \simeq N \otimes \left(\begin{tikzcd} \mathbb{C}[x_1,\ldots,x_n] \arrow[r, bend left, "x_n-g"] & \mathbb{C}[x_1,\ldots,x_n] \arrow[l, bend left, "U_2"] \end{tikzcd} \right) \in \mathrm{MF}(U)$
4. $(i_* \circ i^*) M = \mathrm{Cone}((x_n - g) : M[1] \to M) \in \mathrm{MF}(U)$

Proof. The first proposition is proven in more general setup. See [Pos11] Section 2.1. Second proposition follows from the fact that cokernel commutes with tensor product.

$$Coker(\phi_{10}) \otimes_{\mathbb{C}[x_1,\ldots,x_n]} \mathbb{C}[x_1,\ldots,x_{n-1}] \simeq \mathrm{coker}(\phi_{10}|_{x_n=g}).$$

To prove a third proposition, we should specify Fourier-Mukai kernel of a push-forward functor. Write

$$V(x_1,\ldots,x_{n-1}) - V(y_1,\ldots,y_{n-1}) = \sum_{i}^{n-1} (x_i - y_i) \cdot V_i$$

Define a Koszul-type matrix factorization Γ of

$$V(\vec{x}) - U(\vec{y}) = V(\vec{x}) - \big(V(\vec{y}) + (y_n - g) U_2(\vec{y})\big)$$

as

$$\Gamma := \left(\Lambda^\bullet\langle e_1,\ldots,e_n\rangle, \left(\sum_1^{n-1}(x_i-y_i)i_{e_i} + (y_n-g)i_{e_n} + \sum_1^{n-1} V_i(\cdot\wedge e_i) + U_2(\cdot\wedge V)\right)\right).$$

Under Orlov's equivalence Γ corresponds to a stabilization of a graph $\Gamma_{Y\to X}$. Therefore a Fourier-Mukai functor associated to Γ is a pushforward functor. Notice that

$$-\otimes\Gamma \simeq -\otimes\Delta_V\otimes\left(\begin{tikzcd} \mathbb{C}[x_1,\ldots,x_n] \arrow[r, bend left, "x_n-g"] & \mathbb{C}[x_1,\ldots,x_n] \arrow[l, bend left, "U_2"] \end{tikzcd}\right).$$

where Δ_V is a stabilized diagonal of V. This proves the third proposition.

For the fourth proposition, observe that $i_*\circ i^*M$ goes to

$$\operatorname{coker}\left(\phi_{10}|_{x_n=g} : M^{odd}|_{x_n=g}\to M^{even}|_{x_n=g}\right)$$

under Orlov's equivalence. It is easy to see that the periodic tail of the following double complex realizes the matrix factorization associated to that module.

$$\begin{tikzcd} \cdots \arrow[r, "\phi_{10}"] & M^{even} \arrow[r, "\phi_{01}"] \arrow[d, "x_n-g"] & M^{odd} \arrow[r, "\phi_{10}"] \arrow[d, "x_n-g"] & M^{even} \arrow[d, "x_n-g"] \\ \cdots \arrow[r, "\phi_{10}"] & M^{even} \arrow[r, "\phi_{01}"] & M^{odd} \arrow[r, "\phi_{11}"] & M^{even} \end{tikzcd}$$

This is equal to $\operatorname{Cone}((x_n-g) : M[-1]\to M)$. □

An analogy of a function ring for matrix factorization is its Jacobian ring $\operatorname{Jac}(U)$ and its element acts on a matrix factorization by a multiplication. We get another DG model for $\operatorname{MF}(V)$ as an object inside $\operatorname{MF}(U)$ with vanishing (x_n-g)-action.

Corollary 4.2.2. *Define a DG algebra*

$$\mathscr{B} := \operatorname{Jac}(U)[\epsilon]\Big/\left(\begin{array}{l}\epsilon^2=0\\ d\epsilon = x_n-g\end{array}\right),\quad deg\epsilon=-1$$

Then

$$\mathrm{MF}(V) \simeq \mathscr{B} - mod\,(\mathrm{MF}(U))$$

Proof. Again, this should be a corollary of Barr-Beck-Lurie theorem. In detail, since $U_2 = \partial_{x_n} U$, a multiplication of U_2 is homotopically zero. We see

$$i_* N \simeq \left(\; M[1] \xrightarrow{x_n - g} M \; \right) \simeq M[\epsilon] \Big/ \left(\begin{array}{l} \epsilon^2 = 0 \\ d\epsilon = x_n - g \end{array} \right)$$

for some $M \in \mathrm{MF}(U)$ satisfying $I^* M = N$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{B}}(i_* N_1, i_* N_2) &\simeq \mathrm{Hom}_{\mathscr{B}}(M_1[\epsilon], M_2[\epsilon]) && (4.2.1) \\ &\simeq \mathrm{Hom}_{\mathrm{MF}(U)}(M_1, M_2[\epsilon]) \Big/ \left(\begin{array}{l} \epsilon^2 = 0 \\ d\epsilon = x_n - g \end{array} \right) && (4.2.2) \\ &\simeq \mathrm{Hom}_{\mathrm{MF}(U)}(M_1, (i_* \circ i^*) M_2) && (4.2.3) \\ &\simeq \mathrm{Hom}_{\mathrm{MF}(V)}(N_1, N_2). && (4.2.4) \end{aligned}$$

$\square$

Chapter 5

Equivariant topology of Milnor fiber for invertible curve singularities

In this section, we explain the topology of an invertible curve singularity W. Namely we first describe topology of its Milnor fiber $M_W = W^{-1}(1)$ and their maximal symmetry group G_W. We show in Proposition 5.1.5 that the quotient $[M_W/G_W]$ is homeomorphic to an orbifold sphere with three special points, which are either orbifold points or (orbifold) punctures. We also describe orbifold covering and an action of a deck transformation group in detail.

5.1 Topology of a Milnor fiber

This chapter is mostly borrowed form [Jeo19]. Recall that Milnor fiber is homotopy equivalent to the bouquet of μ-circles where μ is the Milnor number of the singularity.

Lemma 5.1.1. *The weights and Milnor numbers of curve singularities are as follows.*

1. *Weights of $F_{p,q}$ are $(q,p;pq)$. Its Milnor number is $(p-1)(q-1)$.*
2. *Weights of $C_{p,q}$ are $(q,p-1;pq)$. Its Milnor number is $pq-p+1$.*
3. *Weights of $L_{p,q}$ are $(q-1,p-1;pq-1)$. Its Milnor number is pq.*

Proof. Milnor numbers can be easily computed by the following formula.

Theorem 5.1.2 (Milnor-Orlik)**.** *[MO70] Let $f(x_1, \cdots, x_{n+1})$ be the weighted homogeneous polynomial of weights $(w_1, \cdots, w_{n+1}, h)$. Then, it has isolated singularity at the origin whose Milnor number is given by*

$$\mu(0) = (\frac{h}{w_1} - 1) \cdots (\frac{h}{w_{n+1}} - 1)$$

□

Definition 5.1.3. *The maximal diagonal symmetry group G_W of W is defined by*

$$G_W = \{(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2 \mid W(\lambda_1 x, \lambda_2 y) = W(x, y)\}$$

It is easy to check the following.

Lemma 5.1.4. *$G_{F_{p,q}} \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$, $G_{C_{p,q}} \simeq \mathbb{Z}/pq\mathbb{Z}$ and $G_{L_{p,q}} \simeq \mathbb{Z}/(pq-1)\mathbb{Z}$.*

Proof. $G_{F_{p,q}} = \left\{\left(\exp\left(\frac{2k\pi\sqrt{-1}}{p}\right), \exp\left(\frac{2l\pi\sqrt{-1}}{q}\right)\right) \middle| \, 0 \le k \le p-1, 0 \le l \le q-1\right\}$. The generators of $G_{C_{p,q}}$ and $G_{L_{p,q}}$ are (ξ^{-p}, ξ) and (η, η^{-p}) respectively for $\xi = \exp\left(\frac{2\pi\sqrt{-1}}{pq}\right)$, $\eta = \exp\left(\frac{2\pi\sqrt{-1}}{pq-1}\right)$. □

For curve singularities, M_W is given by a (non-compact) Riemann surface. Recall that the boundary of a Milnor fiber is called the link of the singularity which are union of k circles for curve singularities. In our case, we compactify M_W to $\overline{M}_W$ by shrinking each circle of the link to a point. Therefore, $\overline{M}_W \setminus M_W$ consists of k-points. G_W acts on M_W as well as $\overline{M}_W$.

Proposition 5.1.5. *For invertible curve singularities, the genus g, the number of boundary components k of the Milnor fiber M_W is given as follows. Also the* *quotient $[\overline{M}_W/G_W]$ is an orbifold projective line $\mathbb{P}^1_{a,b,c}$:*

$$\begin{aligned}
&\textit{(Fermat)} \quad g = \tfrac{\mu_F+1-d}{2}, k = d, && (a,b,c) = (p,q,\frac{pq}{d}), \ \ d = gcd(p,q)\\
&\textit{(Chain)} \quad g = \tfrac{\mu_C-d}{2}, k = d+1, && (a,b,c) = (pq,q,\frac{pq}{d}), \ \ d = gcd(p-1,q)\\
&\textit{(Loop)} \quad g = \tfrac{\mu_L-1-d}{2}, k = d+2, && (a,b,c) = (pq-1,pq-1,\frac{pq-1}{d}),\\
& && d = gcd(p-1,q-1)
\end{aligned}$$

Here, c vertex for Fermat, a,c-vertices for Chain, a,b,c-vertices for Loop type are punctures.

Proof. It is well-known that the number of boundary components are the same as the number of irreducible factors of W. (Recall that for sufficiently small r and $0 < \epsilon \ll r$, the link$W^{-1}(0) \cap S_r^{2m-1}$ and $W^{-1}(\epsilon) \cap S_r^{2m-1}$ are diffeomorphic and note that each factor of W gives a boundary component for $W^{-1}(0)$). For Fermat type, $x^p + y^q$ factors into d factors for $d = gcd(p,q)$. For Chain type, since $x^p + xy^q = x(x^{p-1} + y^q)$, $C_{p,q}$ has $d+1$ factors with $d = gcd(p-1,q)$. For loop type, since $x^p y + xy^q = xy(x^{p-1} + y^{q-1})$, $L_{p,q}$ has $d+2$ factors with $d = gcd(p-1,q-1)$.

To compute the genus, note that M_W is obtained by removing k punctures from $\overline{M}_W$. Hence, Euler charactoristic $\mathscr{E}(M_W) = \mathscr{E}(\overline{M}_W) - k$. But M_f has the homotopy type of bouquet of μ-circles for the Milnor number μ, and its Euler charactoristic $\mathscr{E}(M_W) = 1-\mu$. Therefore, the genus of $\overline{M}_W$ (and hence M_W) is obtained from $2-2g-k = 1-\mu$ or $g = (\mu+1-k)/2$.

Now, to find the quotient orbifold $[\overline{M}_W/G_W]$, we first find there are exactly three orbits (of G_W) with non-trivial stabilizer in $\overline{M}_W$ and show that the quotient has genus zero using orbifold-Euler-characteristic. We will use the fact that $\mathscr{E}(\overline{M}_W)/|G_W|$ equal the orbifold Euler-characteristic of $[\overline{M}_W/G_W]$.

Let us consider the Fermat case. Orbits of $[(0,1)]$ and $[(1,0)]$ gives two singular orbits of $\mathbb{Z}/p \oplus \mathbb{Z}/q$-action on M_F. They have stabilizers $\mathbb{Z}/p, \mathbb{Z}/q$ respectively. For $d = gcd(p,q)$, we have d punctures and $\mathbb{Z}/p \oplus \mathbb{Z}/q$ acts transitively on them. So the quotient has three orbifold points $(a,b,c) = (\mathbb{Z}/p, \mathbb{Z}/q, \mathbb{Z}/(pq/d))$.

To see that the quotient is $\mathbb{P}^1_{a,b,c}$,

$$\mathscr{E}(\overline{M}_W) = 2-2g = k-1-\mu = d-1-(p-1)(q-1)$$

Note that it equals $|G|\cdot\mathscr{E}_{orb}(\mathbb{P}^1_{a,b,c})$ which is

$$(pq)\cdot(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1) = pq\cdot(\frac{1}{p}+\frac{1}{q}+\frac{d}{pq}-1)$$

This proves the claim for the Fermat case.

The other cases are similar. For the chain case, the orbit of $(1,0)$ has stabilizer $\mathbb{Z}/p$. The other two orbifold points come from punctures. Note that $C_{p,q}$ is a product of x and $x^{p-1}+y^q$. It is easy to see that G_W action preserves each branches $x=0$ as well as $x^{p-1}+y^q=0$. Hence the puncture corresponding to the branch x has the full group G_f as a stabilizer and the other d punctures (for the factors of $x^{p-1}+y^q=0$ with $d=gcd(p-1,q)$) are acted by G_f in a transitive way. Therefore, the orbifold point has stabilizer $\mathbb{Z}/(pq/d)$. For the loop type, M_W has no fixed points of G_W-action, and the punctures for factors $x, y, x^{p-1}+y^{q-1}$ form three orbits with stabilizer $\mathbb{Z}/(pq-1), \mathbb{Z}/(pq-1), \mathbb{Z}/((pq-1)/d)$. This finishes the proof. □

5.2 Orbifold covering

In the previous section, we observed that G_W acts on the Milnor fiber M_W to produce the following regular orbifold covering

$$\overline{M}_W \to \mathbb{P}^1_{a,b,c}.$$

Given a Riemann surface, there can be two non-equivalent group actions with isomorphic quotient space(see Broughton [Bro91] for example). Hence, to determine the G_f-action explicitly, we find an explicit group homomorphism

$$\phi : \pi_1^{orb}\left(\mathbb{P}^1_{a,b,c}\right) \to G_W \tag{5.2.1}$$

from orbifold fundamental group onto the symmetry group G_W. For the kernel $\Gamma = \mathrm{Ker}(\phi)$, $\overline{M}_W$ is an orbifold covering of $\mathbb{P}^1_{a,b,c}$ corresponding to the kernel Γ with deck transformation group G_W.

We use the following presentation of the orbifold fundamental group of $\mathbb{P}^1_{a,b,c}$

$$\pi_1^{orb}\left(\mathbb{P}^1_{a,b,c}\right) = \left\langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^a = \gamma_2^b = \gamma_3^c = \gamma_1\gamma_1\gamma_3 = 1 \right\rangle \tag{5.2.2}$$

Here, γ_1 is a small loop going around $0 \in \mathbb{P}^1$. γ_2 is for $1 \in \mathbb{P}^1$ and γ_3 is for $\infty \in \mathbb{P}^1$. Later on, this presentation will serve as an additional grading on a Floer theory.

Proposition 5.2.1. *The homomorphism* (5.2.1) *is given as follows.*

1. *(Fermat) For the covering* $M_{F_{p,q}} \to \mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}$, *we have*

$$\phi(\gamma_1) = (1,0),\ \phi(\gamma_2) = (0,1),\ \phi(\gamma_3) = (-1,-1) \in \mathbb{Z}/p \oplus \mathbb{Z}/q$$

2. *(Chain) For the covering* $M_{C_{p,q}} \to \mathbb{P}^1_{pq,q,\frac{pq}{\mathrm{gcd}(p-1,q)}}$, *we have*

$$\phi(\gamma_1) = 1,\ \phi(\gamma_2) = -p,\ \phi(\gamma_3) = p-1 \in \mathbb{Z}/pq$$

3. *(Loop) For the covering* $M_{L_{p,q}} \to \mathbb{P}^1_{pq-1,pq-1,\frac{pq}{gcd(p-1,q-1)}}$, *we have*

$$\phi(\gamma_1) = 1,\ \phi(\gamma_2) = -p,\ \phi(\gamma_3) = p-1 \in \mathbb{Z}/(pq-1)$$

Let us give the proof in each cases separately.

5.2.1 Fermat type $F_{p,q}$

$M_{F_{p,q}}$ is a locus of an equation $x^p + y^q = 1$. We regard them as a Riemann surface of a multivalued function

$$y = (1-x^p)^{\frac{1}{q}}, \quad k = 0, \ldots q-1$$

with q branch points $x_i = e^{\frac{2k\pi i}{q}}$ (for $i = 0, \cdots, q-1$). We connect branch point x_i with ∞ by a ray $\{re^{\frac{2k\pi i}{q}} \mid r \geq 1\}$. With this branch cut, $M_{F_{p,q}}$ is a q sheeted covering of a complex plane $\mathbb{C}$.

A fundamental domain of the quotient is the following "pizza" shape domain.

$$\left\{x = re^{\theta} | 0 \leq r,\ 0 \leq \theta \leq \frac{2\pi}{p}\right\} \tag{5.2.3}$$

There are three distinguished paths $\gamma_i : [0,1] \to \mathbb{C}$.

- $\gamma_1(t) = \epsilon \cdot e^{\frac{2\pi i t}{p}}$, $(0 < \epsilon \ll 1)$, a small path around the origin.
- $\gamma_2(t) = 1 + \epsilon \cdot e^{2\pi i t}$, $(0 < \epsilon \ll 1)$ a small circle around the branch points.
- $\gamma_3(t) = Re^{\frac{-2\pi i t}{p}}$, $(R \gg 1)$ a boundary cirle with opposite orientation.

These are orbifold loops that correspond to generators of $\pi_1\left(\mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}\right)$ in 5.2.2.

Let us find the the homomorphism (5.2.1) for the Fermat case. Recall that we realize $F_{p,q}$ as a q sheeted covering of $\mathbb{C}$. Label those sheets by natural numbers from 1 to q so that the crossing branch cuts increases the label number by $+1$. Each sheet has p copies of the fundamental domain. We put the label i_j on the following copies of it;

$$\left\{x = re^{\theta} | 0 \leq r,\ \frac{2(j-1)\pi}{p} \leq \theta \leq \frac{2j\pi}{p}\right\} \subset i\text{th sheet.}$$

In this setup, we can write down the monodromy representation of the fundamental group to the group of permutation of the set of labels $\{i_j | 0 \leq i \leq q,\ \ 0 \leq j \leq p\}$.

$$\begin{aligned}
\phi : \pi_1\left(\mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}\right) &\to S_{pq} \\
\gamma_1 &\mapsto (1_1, 1_2, \ldots, 1_p)(2_1, 2_2, \ldots, 2_p)\cdots(q_1, q_2, \ldots, q_p) \\
\gamma_2 &\mapsto (1_1, 2_1, \ldots, q_1)(1_2, 2_2, \ldots, q_2)\cdots(1_p, 2_p, \ldots, q_p) \\
\gamma_3 &\mapsto (\gamma_1 \circ \gamma_2)^{-1}
\end{aligned}$$

The image of this representation is generated by γ_1 and γ_2, isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/q$. Moreover, it is compatible to the diagoal symmetry group action.

- γ_1 is a rotation of each sheets by $\frac{2\pi}{p}$. It corrsponds to a diagonal action $x \to e^{\frac{2i\pi}{p}} \cdot x, \ \ y \to y$.
- γ_2 is a rotation of each sheets by $\frac{2\pi}{q}$ so it corrsponds to a diagonal action $x \to x, \ \ y \to e^{\frac{2i\pi}{p}} \cdot y$

5.2.2 Chain type $C_{p,q}$

$M_{C_{p,q}}$ is a locus of an equation $x^p + xy^q = 1$. We regard them as a Riemann surface of a multivalued meromorphic function

$$y = \left(\frac{1-x^p}{x}\right)^{\frac{1}{q}}, \qquad k = 0, \dots q-1.$$

This function has a q zero branch points $x = e^{\frac{2k\pi i}{q}}$ and a single pole branch point $x = 0$.

We connect each branch points with ∞ by rays as before. Also, we overlap a ray from a pole $x = 0$ and a ray from a zero $x = 1$. Because they are coming out of different sources, they cancel each others on the overlap. With this choice of branch cuts, $M_{C_{p,q}}$ is a q sheeted covering of $\mathbb{C}^*$.

A fundamental domain of the quotient $M_{C_{p,q}}/G_{C_{p,q}}$ can be taken as the same domain (5.2.3) but in $\mathbb{C}^*$ and orbifold loops $\gamma_1, \gamma_2, \gamma_3$ are the same as in the Fermat cases. Hence γ_1, γ_3 are the loops around the orbifold punctures.

Due to the branch cut along the line segment $[0,1]$ on the real axis, monodromy representation is different from the Fermat cases. It is not hard to see that we get the following symmetric group representation

$$\begin{aligned}
\phi : \pi_1\left(\mathbb{P}^1_{pq,q,\frac{pq}{gcd p-1,q}}\right) &\to S_{pq} \\
\gamma_1 &\mapsto (1_1, 1_2, \dots, 1_p, q_1, q_2, \dots, 3_p, 2_1, 2_2, \dots, 2_p) \\
\gamma_2 &\mapsto (1_1, 2_1, \dots, q_1)(1_2, 2_2, \dots, q_2) \cdots (1_p, 2_p, \dots, q_p) \\
\gamma_3 &\mapsto (\gamma_1 \circ \gamma_2)^{-1}.
\end{aligned}$$

Unlike the Fermat case, $\phi(\gamma_1)$ generates $\phi(\gamma_2)$ by the relation $\phi(\gamma_2) = \phi(\gamma_1)^{-p}$. Therefore the image of ϕ is generated by γ_1, isomorphic to $\mathbb{Z}/pq$. Notice that

$\phi(\gamma_1)$ rotates each sheet by $\frac{2\pi}{p}$ and change the label of a sheet by $+1$ if you apply it by $-p$ times. It corresponds to a diagonal action

$$x \to e^{\frac{2\pi i}{p}} \cdot x, \quad y \to e^{\frac{-2\pi i}{pq}} \cdot y$$

which is a generator of the $\mathbb{Z}/pq$-action.

5.2.3 Loop type $L_{p,q}$

$M_{L_{p,q}}$ is a locus of an equation $x^p y + xy^q = 1$. As we can not realize $L_{p,q}$ as a Riemann surface of a single function, we work with the following parametrization by $z \in \mathbb{C}$

$$x = \left(\frac{z^q}{1-z}\right)^{\frac{1}{pq-1}}, \quad y = \left(\frac{(1-z)^p}{z}\right)^{\frac{1}{pq-1}}, \qquad k = 0, \dots pq-2$$

with two branch points $z = 0, 1$.

The point $z = 0$ is an order q zero for x and order 1 pole for y. Likewise, the point $z = 1$ is an order 1 pole for x and order p zero for y. We connect these two with ∞ by half lines and let the one from the origin overlaps the one from $z = 1$. Although it is a Riemann sureface of two multivalued functions rather then a single one, we can still $M_{L_{p,q}}$ is now a $pq-1$ sheeted covering of a z-plane $\mathbb{C} \setminus \{0, 1\}$ as follows.

A fundamental domain is the whole z-plane minus two points $z = 0, 1$. The three distinguished paths are

- $\gamma_1(t) = \epsilon \cdot e^{2\pi i t}$, $(0 < \epsilon \ll 1)$, a small circle around $z = 0$.
- $\gamma_2(t) = 1 + \epsilon \cdot e^{2\pi i t}$, $(0 < \epsilon \ll 1)$ a small circle around $z = 1$.
- $\gamma_3(t) = Re^{-2\pi i t}$, $(R \gg 1)$ a boundary circle with opposite orientation.

Let us compute the monodromy representation. Whenever we cross brach cuts inside z-plane, we change a covering sheet for x and y both. Each of them has $pq-1$ possibilities, so there are $(pq-1) \times (pq-1)$ different sheets. Let's label them by (i, j), $i, j = 1, \dots, pq-1$. But we don't need all of them because

π_1-orbit of $(1,1)$ consists of only $pq-1$ sheets among them. The monodromy representation of π_1^{orb} is written as

$$\begin{aligned}\phi:\pi_1^{orb}\Big(\mathbb{P}^1_{pq-1,pq-1,\frac{pq}{gcd(p-1,q-1)}}\Big) &\to S_{pq-1}\times S_{pq-1}\\ \gamma_1 &\mapsto (+q,-1):(a,b)\to(a+q,b-1)\\ \gamma_2 &\mapsto (-1,p):(a,b)\to(a-1,b+p)\\ \gamma_3 &\mapsto (\gamma_1\circ\gamma_2)^{-1}\end{aligned}$$

Since $\phi(\gamma_2)=\phi(\gamma_1)^{-p}$, the image of this representation is generated by γ_1 and isomorphic to $\mathbb{Z}/pq-1$. Notice that γ_1 corresponds to the following element

$$x\to e^{\frac{2q\pi i}{pq-1}}\cdot x,\quad y\to e^{\frac{-2\pi i}{pq-1}}\cdot y$$

of maximal diagonal symmetry group.

5.3 Equivariant tessellation of Milnor fibers

This chapter is a work of [Jeo19]. Using the equator of $\mathbb{P}^1_{a,b,c}$ containing three orbifold points, we can divide the orbi-sphere into two cells. From the orbifold covering $\overline{M}_W\to\mathbb{P}^1_{a,b,c}$, and considering lifts of these two cells, we obtain a tessellation of Milnor fibers of invertible curve singularities. In this section, we give a combinatorial description of the tessellation of $\overline{M}_W$ as well as G_W-action on it.

Consider a $2m$-gon whose boundary edges are labelled by $a_1,\cdots,a_{2m}$ ordered and oriented in a counterclockwise way. We say edges are identified as $\pm(2p-1)$ pattern if a_{2k} and $(a_{2k-(2p-1)})^{op}$ are identified, and a_{2k-1} and $(a_{2k-2p})^{op}$ are identified for any k. (Here indices are modulo $2m$, and a^{op} is the orientation reversal of the edge). Note that even and odd numbered edges play different roles. See Figure 5.3 (B) for 16-gon identified with ± 7 pattern.

Theorem 5.3.1. *Compactified Milnor fiber $\overline{M}_W$ and G_W on it are explicitly described as follows*

1. *(Fermat) $\overline{M}_{F_{p,q}}$ is given by $(2pq-2q)$-gon with edges identified as $\pm(2p-1)$*

pattern. An odd numbered edge corresponds to an oriented path from c-vertex to b-vertex in the quotient.

2. *(Chain) $\overline{M}_{C_{p,q}}$ is given by $(2pq)$-gon with edges identified as $\pm(2p-1)$ pattern. An odd numbered edge corresponds to an oriented path from b-vertex to c-vertex in the quotient.*

3. *(Loop) $\overline{M}_{L_{p,q}}$ is given by $2(pq-1)$-gon with edges identified as $\pm(2p-1)$ pattern. An odd numbered edge corresponds to an oriented path from b-vertex to c-vertex in the quotient.*

Proof. Recall that we have $[\overline{M}_W/G_W] = \mathbb{P}^1_{a,b,c}$ from Proposition 5.1.5. Let H be the universal cover of $\overline{M}_f$ or equivalently that of $\mathbb{P}^1_{a,b,c}$. We have $\pi_1^{orb}(\mathbb{P}_{a,b,c})$ action on H. Let F be a fundamental domain in H for this action as in the Figure 5.1 where the angle is measured in S^2 or $\mathbb{R}^2$ or $\mathbb{H}$ depending on the universal cover. Here x_1, x_2, x_3 project down to a, b, c orbifold points and at x_1, x_2 we have the full cone angle but the cone angle for x_3 is divided into half for x_3 and x_3'. Also, we will use Proposition 5.2.1 which describes the relation between generators $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1^{orb}(\mathbb{P}_{a,b,c})$ and G_f.

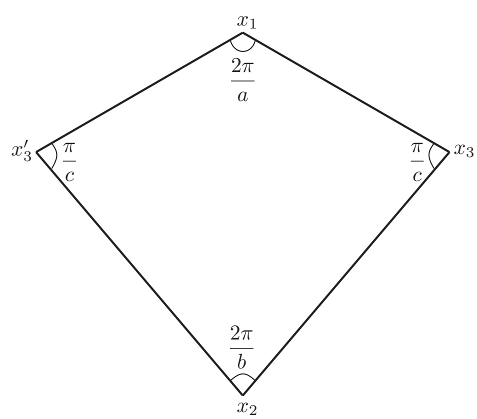


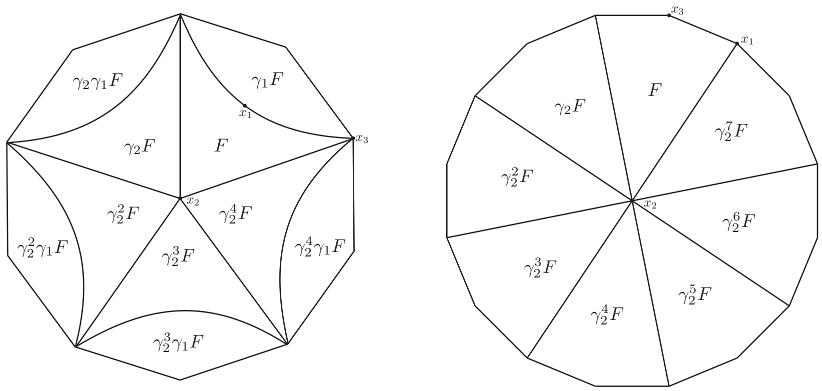
Figure 5.1: Fundamental domain of $\mathbb{P}^1_{a,b,c}$ in $\mathbb{H}$

For the Fermat case, consider $\gamma_1, \gamma_2 \in \pi_1^{orb}(\mathbb{P}_{a,b,c})$ and collect the following $p \times q$ copies of F to define a polygon

$$P := \left\{ \gamma_2^i \gamma_1^j F \mid 0 \le i \le p-1, 0 \le j \le q-1 \right\}.$$

First, P is a fundamental domain of $\overline{M}_W$ since $G_W = \mathbb{Z}/p \times \mathbb{Z}/q$ and $\phi(\gamma_1) = (1,0), \phi(\gamma_2) = (0,1)$ are the generators of G_W by Proposition 5.2.1.

Also, one can check that P is a $(2pq-2q)$-gon in the following way. First, $\{\gamma_1^j F\}$ for $j = 0, \cdots, p-1$ can be glued counter-clockwise way around the vertex x_1 of Figure 5.1 to form a $2p$-gon, say Q. Then, by applying $\{\gamma_2^i\}$ for $i = 0, 1, \cdots, q-1$ to Q, we get q-copies of Q glued around the vertex x_2 to form a $(2pq-2q)$-gon and this is exactly P. Because $2q$ edges meeting the vertex x_2 become interior edges, number of boundary edges decrease by $2q$ from $2pq$. See Figure 5.2 (A) for the case of $\overline{M}_{F_{2,5}}$, where Q is given by the union of F and $\gamma_1 F$ and P is the 10-gon.



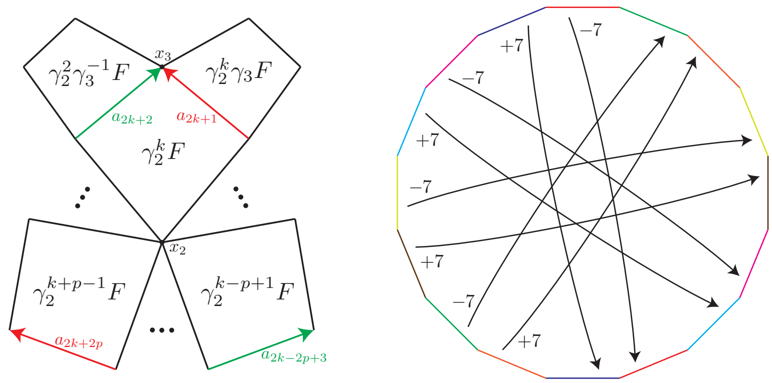
(a) $\overline{M}_{F_{2,5}}$ case

(b) $\overline{M}_{C_{4,2}}$ case

Figure 5.2: Tessellations of A_4 and D_5 singularities

Note that P has an obvious induced $\mathbb{Z}/p \times \mathbb{Z}/q$-action. Namely, $\mathbb{Z}/p$-action is the $\frac{2\pi}{p}$ rotation around the center of P, and $\mathbb{Z}/q$-action is the $\frac{2\pi}{q}$ rotation of every copy of Q around their centers. In fact, for the edge e which is given by the intersection $(\gamma_2^i Q) \cap (\gamma_2^{i+1} Q)$, action by $\mathbb{Z}/q$ on e will depend on whether we interpret e as an element of $(\gamma_2^i Q)$ or $(\gamma_2^{i+1} Q)$. But the boundary identification of P is exactly the relations that make two $\mathbb{Z}/q$-actions on e to agree with each other. One can check that it gives $\pm(2p-1)$ identification on ∂P. This proves the proposition for the Fermat case.

Next, let us discuss the chain type. Recall that we have $G_{C_{p,q}} = \mathbb{Z}/pq$, and $\phi(\gamma_1) = 1 \in \mathbb{Z}/pq$ is the generator. We take the following pq-copies of F to de-



(a) Boundary identification

(b) ±7 pattern

Figure 5.3: Chain cases

fine a $2pq$-gon:

$$P := \left\{\gamma_1^i F \mid 0 \le i \le pq-1\right\},$$

which is a fundamental domain for the Milnor fiber. To find the boundary identification, we consider additional tiles next to P. To see how two boundary edges from $\gamma_1^k F$ are identified to the remaining edges, we consider the addition rotation action around vertex for x_2. Namely, consider $\gamma_2^{-1}\gamma_1^k F$ and $\gamma_2\gamma_1^k F$. Since $\phi(\gamma_2) = -p$, we have $\phi(\gamma_2^{-1}\gamma_1^k) = \phi(\gamma^{p+k})$. Therefore, $\gamma_2^{-1}\gamma_1^k F$ can be identified with $\gamma^{p+k}F$ as a tile in the Milnor fiber. From this, we can deduce that x_2x_3 edge of $\gamma_1^k F$ should be identified with x_2x_3' edge of $\gamma_1^{k+p}F$. See Figure 5.3 (A). In terms of edges of P, this is $+(2p-1)$ identification. From the same argument for $\gamma_2\gamma_1^k F$, we find that x_2x_3' edge of $\gamma_1^k F$ should be identified with $-(2p-1)$ pattern. This proves the chain cases.

For the loop type, we can proceed similarly as in the chain case. We take

$$P := \{\gamma_3^i F \,|\, 0 \le i \le pq-2\}$$

and we get the same identification as in the chain case from Proposition 5.2.1. □

Remark 5.3.2. *We remark that $\overline{M}_W$ is a sphere for $F_{2,2}$, and is a torus for $F_{3,2}, F_{4,2}$, $C_{2,2}, C_{3,2}, L_{2,2}$ and a higher genus surface for the rest of the cases. For the last case, universal cover can be taken as the hyperbolic plane $\mathbb{H}$ and there exists a*

Fuchsian group Γ *such that* $\mathbb{H}/\Gamma \simeq \overline{M}_W$. *Furthermore, a finite group G acts on M if and only if there exist a Fuchsian group* Γ' *and a surjective homomorphism* $\phi : \Gamma' \to G_W$ *with torsion-free kernel* Γ *such that* $M \simeq \mathbb{H}/\Gamma$ *and* $M/G \simeq \mathbb{H}/\Gamma'$.

Chapter 6

Equivariant Floer theory of a Milnor fiber

In this section, we apply our general Floer theories to an orbifold quotient $[M_W/G_W]$ of an invertible curve singularities.

6.1 Hamiltonian

$[M_W/G_W]$ is a three-punctured sphere with orbifold/punctures at $\{0,1,\infty\} \in \mathbb{P}^1$. At each point, we are using one of the following orbifold/puncture chart with G-equivarant ;

- a disc chart
$$\left(D^2, \mathbb{Z}/n, dx \wedge dy\right)$$
with coordrdinate $w = x + iy$. $\mathbb{Z}/n$ acts on D by rotation.

- a puncture chart
$$\left(S^1 \times [0,\infty), \mathbb{Z}/n, dr \wedge d\theta\right)$$
with coordinate system $w = e^{r+\theta i}$. $\mathbb{Z}/n$ acts on $S^1 \times [0,\infty)$ by rotation on θ coordinate.

To keep an orbifold information of disc charts, we further restrict our positive Hamiltonian H to be G_W-equivariant and of the following form;

- outside puncture charts, H is Morse and C^2-small enough so that it does not have a time-1 periodic orbit outside of a cylinder chart.
- at a puncture chart, H is quadratic w.r.t r:

$$H = ar^2 + b, \quad (a > 0).$$

- at a disc chart, H is a function of r and have the unique Morse minimum at $0 \in D$. For example,

$$H = \epsilon - \frac{\epsilon}{2} e^{-r^2}, \quad (0 < \epsilon << 1)$$

This class of hamiltonian is $\mathbb{Z}/n$- equivariant and its pull-back is still quadratic at the end. We sometimes using a "uniformization"

$$x = w^n$$

to describe a complex neighborhood of an orbifold point. Be aware, under this coordinate transform, a pull-back of quadratic Hamiltonians is no longer quadratic. Whenever we use such notation we implicitly replace the neighborhood of the origin to above disc/cylinder charts.

6.2 Ω- and H^1-grading

We describe two grading systems on the Floer theory on $[M_W/G_W]$. It is a slight generalization of [Sei11].

An assumption $c_1(M) = 0$ was crucial for $\mathbb{Z}$ gradings on symplectic cohomology or wrapped Floer cohomology. Unfortunately, an orbifold canonical bundle $K_{[M_W/G]}$ is never trivial. Therefore it is reasonable to expect that the Floer theory on $[M_W/G]$ cannot have a compatible $\mathbb{Z}$-grading. As a partial remedy, we use a holomorphic volume form with specific poles. Up to constant, there is a unique holomorphic volume form Ω on $\mathbb{P}^1$ with poles of order one at $0, 1 \in \mathbb{P}^1$.

This choice provides a trivialization of a tangent bundle $T_{[M_W/G]}$ away from $0, 1$ and ∞. Because of our choice of Hamiltonians, its time-1 orbits are always disjoint from $0, 1, \infty \in \mathbb{P}^1$ after adding a small time-dependent perturbation term. Therefore, each hamiltonian orbits still carries an honest cohomological conley-Zehnder index. we use this integer as a degree.

We can put a grading on a Lagrangians and Hamiltonian chords between them in a similar fashion. For a Lagrangian L which is oriented and away from $0, 1, \infty \in \mathbb{P}^1$, we get a phase map w.r.t Ω;

$$\overline{\phi}_L : L \to S^1 \tag{6.2.1}$$

$$\overline{\phi}_L(x) = \frac{\Omega(X)}{|\Omega(X)|} \tag{6.2.2}$$

where X is a nonvanishing vector field on TL pointing positive direction.

Definition 6.2.1. *An Ω-grading on L is a choice of lift*

$$\phi_L : L \to \mathbb{R}$$

of a phase map $\overline{\phi}$. An Ω-graded Lagrangian L is a Lagrangian submanifold with a specific choice of Ω- grading ϕ_L.

For any time-1 hamiltonian chords $a \in \chi(L_0, L_1)$ between graded Lagrangian submanifold, there is a unique homotopy class of Lagrangian path from $T_{L_0, a(0)}$ to $T_{L_1, a(1)}$ compatible to the gradings. The absolute Maslov index $\mu_M(a)$ is now well defined, and we use it as a degree of a.

A discrepancy occurs when we consider a moduli space of discs. A standard index formula starts to read an intersection number of a holomorphic maps and pole divisor of Ω. Let

$$\overline{\mathcal{M}}_{m;n,1;[u]}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$$

be a sub-moduli space of $\overline{\mathcal{M}}_{m;n,1;u}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ whose relative class is $[u]$. Then a standard index formula is now read

$$\begin{aligned}\dim_{\mathbb{R}} \overline{\mathcal{M}}_{m;n,1;[u]}(\gamma_1,\dots,\gamma_m;a_1,\dots,a_n,a_0) = (2m+n-2) \\ + \deg a_0 - \sum_{i=1}^{n} \deg a_i - \sum_{j=1}^{n} \deg \gamma_j \\ + 2\big(\deg(u,0)+\deg(u,1)\big)\end{aligned}$$

The dimension of our moduli space may differ in even numbers. It breaks a $\mathbb{Z}$-grading into $\mathbb{Z}/2$-grading.

Meanwhile, there is a topological grading coming from an orbifold cohomology. Recall we use a notation $\gamma_0, \gamma_1, \gamma_\infty$ to denote a homotopy class of loops winding orbifold point $0, 1$ or ∞ respectively. We use the same notation for corresponding homology class. We get

$$H^1_{orb}([M_W/G_W]) \simeq \begin{cases} \mathbb{Z}\langle\gamma_1,\gamma_2,\gamma_3\rangle/\{p\gamma_1 = q\gamma_2 = \gamma_1+\gamma_2+\gamma_3 = 0\} & (W = x^p + y^q) \\ \mathbb{Z}\langle\gamma_1,\gamma_2,\gamma_3\rangle/\{q\gamma_1 = \gamma_1+\gamma_2+\gamma_3 = 0\} & (W = x^p + xy^q) \\ \mathbb{Z}\langle\gamma_1,\gamma_2,\gamma_3\rangle/\{\gamma_1+\gamma_2+\gamma_3 = 0\} & (W = x^p y + xy^q) \end{cases}$$

Notice that any symplectic cochains, including Morse critical point, can be considered as an element of H^1_{orb}. Moreover, if the Lagrangian submanifold L is simply connected, elements of $CW^\bullet(L,L)$ can also be labeled by H^1_{orb}. Let's call it an *H^1-grading*. The Floer theoretic operation uses pseudo-holomorphic curves whose homological boundary is a difference of homology class of an output and inputs. Therefore, we have

Lemma 6.2.2. *A pseudo-holomorphic curve operation is homogeneous with respect to an H^1-grading.*

6.3 Orbifold wrapped Fukaya category

For a definition of equivariant Floer cohomology, we follow [Sei11] closely.

The collection of Lagrangians $\mathcal{W}$ we consider are Lagrangians $L \in [M_W/G_W]$ such that

- it is conical at the end;

- it carries a grading and spin structure (=an additional real line bundle);
- it is away from a singular locus $\{0,1,\infty\}$.
- its G_W-orbit $\{g \cdot \tilde{L} : g \in G_W\}$ of a lift $\tilde{L}$ intersect transversally to each other only finitely many times;

For two such Lagrangians $L_0, L_1 \in \mathscr{W}$, *a G_W-equivariant Floer cochain complex* is defined by

$$CW^{G_W,\bullet}(L_0,L_1) := \left(\bigoplus_{g,h \in G_W} CW^\bullet \left(g \cdot \widetilde{L_0}, h \cdot \widetilde{L_1} \right) \right)^{G_W}$$

where $\widetilde{L_i}$ is a lift of L_i. If there is no confusion, we omit a group notation G_W and simply write $CW^\bullet(L_0,L_1)$. In a similar fashion,

Definition 6.3.1. *An orbifold wrapped Fukaya category*

$$\mathscr{W}\mathscr{F}([M_W/G_W])$$

consists of;

1. *a set of objects $\mathscr{W}$;*
2. *space of morphisms are $CW^{G_W,\bullet}(L_0,L_1)$, graded by the parity of* $\deg a$*;*
3. *an A_∞ structure map is a G-invariant part of the m_k-operation of $\mathscr{W}\mathscr{F}(M_W)$.*

Remark 6.3.2. *As noticed in [Sei11], an explicit perturbation scheme extends to the equivariant case without any serious problem. Such perturbation data are inhomogeneous terms of the pseudo-holomorphic curve equation which vary on the* ***domain*** *of the curve instead of the target. The group G acts only on the target space M. Therefore we have enough freedom to extend perturbation data in an equivariant way.*

Equivariant wrapped Floer cohomology is indeed a Floer cohomology of an orbifold. We have a direct sum decomposition

$$\bigoplus_{h\in G_W} CW^\bullet(L_0,L_1)^h \simeq CW^\bullet(L_0,L_1) \tag{6.3.1}$$

$$\gamma \mapsto \sum_{g\in G_W} g(\gamma), \tag{6.3.2}$$

$$CW^\bullet(L_0,L_1)^h := CW^\bullet\left(\widetilde{L_0},\ h\cdot\widetilde{L_1}\right). \tag{6.3.3}$$

Therefore, each chord carries an information of an 'arrow' of an orbifold $[M_W/G_W]$. m_k operation respects this additional index, which means it restricts to

$$m_k : \bigotimes_{g_i\in G} CW^\bullet(L_{i-1},L_i)^{g_i} \to CW^\bullet(L_0,L_k)^{\prod_i g_i}$$

Conversely, we have an action of a character group $\hat{G}_W = \mathrm{Hom}(G_W,\mathbb{C}^*)$ of G on the LHS given by

$$\phi(\gamma) := \phi(h)\cdot\gamma \ \text{ when } \ \gamma\in CW^\bullet\left(\widetilde{L_0},\ h\cdot\widetilde{L_1}\right).$$

It is clear from the definition that A_∞ operation is equivariant with respect to this action, and we obtain

$$\left(\bigoplus_{g,h\in G_W} CW^\bullet\left(g\cdot\widetilde{L_0}, h\cdot\widetilde{L_1}\right)\right) \simeq CW^\bullet(L_0,L_1)\ltimes\hat{G}.$$

Notice that $\hat{G}_W = G_{W^T}$, a Berglund-Hübsch dual group of G_W.

A holomorphic disc $u : S\to M_W$ defining A_∞ structure can be considered a **smooth** holomorphic orbi-discs

$$\overline{u} : S \to [M_W/G_W]$$

ramified at orbifold points accordingly. Conversely, because S has a vanishing orbifold fundamental group, all smooth holomorphic orbi-discs lifts to M_W in a G_W-equivariant manner. Therefore,

Fukaya category of $[M_W/G_W] \leftrightarrow G_W$-equivariant Fukaya category of M_W.

6.4 Orbifold symplectic cohomology

We define an orbifold version of symplectic cohomology for this particular case.

Recall our autonomous Hamiltonian H has orbifold points as its Morse minimum. We also restrict a class of time-dependent perturbation. Whenever we add C^2 small, S^1-dependent function F to autonomous Hamiltonian H, we assume $F = 0$ in a sufficiently small disc chart so that orbifold points is still a Morse critical point of $H+F$.

We define *an orbifold symplectic cochain complex* as

$$CH^\bullet([M_W/G_W]) := \bigoplus_{\gamma\in\mathcal{O}} \mathbb{C}\cdot\gamma$$

where

$$\mathcal{O} := \mathcal{O}(H+F)$$

is a time-1 orbits of hamiltonian function H perturbed by F. By definition, a space of orbifold loops is again an orbifold

$$\mathcal{L}([M_W/G_W]) = \left\{(g,\gamma) \mid g\cdot\gamma(0) = \gamma(1)\right\}.$$

where $h \in G$ acts by

$$h\cdot(g,\gamma) = (hgh^{-1}, h\gamma(t)).$$

By our choice of Hamiltonians, an element of $\mathcal{O}$ falls into one of three types;

- **Morse critical point** of $H+F$ without isotropy group.
- **twisted sectors** represented by a constant loop at the origin of each disc chart
$$(\xi,0) \in \mathcal{L}([D/(\mathbb{Z}/n)]), \quad \xi \in \mathbb{Z}/n;$$
- **Hamiltonian chords** at the end. They are locally represented by a small perturbation of
$$\gamma_k(t) = \left(e^{\frac{2\pi k i t}{n}}, k\right) \subset (S^1\times[1,\infty), \mathbb{Z}/n).$$

It is $\mathbb{Z}/2$-graded by the parity of a $\deg\gamma$. A degree of Morse critical point and

Hamiltonian chords is its cohomological Conley-Zehnder index as usual. We put a degree of twisted sectors as zero.

Remark 6.4.1. *In the Chen-Ruan cohomology theory, a degree of twisted sector* $(\xi,0)$ *is shifted by a rational number* $\frac{2|\xi|}{n}$. *This is because the action of* ξ *on the orbifold tangent bundle is not trivial. In our case, a holomorphic volume form* Ω *we choose has a pole of order one at the orbifold point, so the action of* ξ *is trivial on it. Therefore a cohomological Conley-Zehnder index of* $(\xi,0)$ *is zero. An effect of an isotropy group is absorbed by* Ω.

Its differential d_{CH} is defined as

$$\mathrm{d}_{CH}(\gamma_1) = (-1)^{\deg\gamma}\mathbf{F}_{1,1;0}(\gamma_1).$$

We also define a pair-of-pants product by

$$\gamma_1\cdot\gamma_2 := (-1)^{\deg\gamma_1}\mathbf{F}_{2,1;0}(\gamma_1,\gamma_2)$$

We should point out that a family of smooth pseudo-holomorphic curves may contains an orbifold nodal point. (We avoided this issue in the definition of Fukaya category). Fortunately, we can rule out those contribution in this case.

Lemma 6.4.2. $d_{CH}^2 = 0$. *The product structure induces a ring structure on* d_{CH}-*cohomology.*

Proof. It is enough to show that orbifold nodal degeneration does not affect a standard analysis of codimension 1 boundary strata of moduli spaces. A local model of such degeneration is given by a family

$$z_1 z_2 : \left[\mathbb{C}^2/(\mathbb{Z}/n)\right] \to \mathbb{C}$$

Here, a group $\mathbb{Z}/n$ acts on $\mathbb{C}^2$ by $(z_1,z_2)\to(\xi\cdot z_1,\xi^{-1}\cdot z_2)$ with ξ an n-th root of unity. Generic fibers are a smooth cylinder while the zero fiber is an orbifold nodal curve. From this local model, we conclude that orbifold nodal degeneration happens inside a codimension 2 boundary strata of a moduli space of domains. It does not appear in a codimension 1 boundary strata of $\overline{\mathcal{M}}_{1,1;0}(\gamma_1,\gamma_0)$ and $\overline{\mathcal{M}}_{2,1;0}(\gamma_1,\gamma_0)$ if we choose a generic almost complex structure. $\square$

We call a cohomology group an orbifold symplectic cohomology, denoted by

$$SH^\bullet([M_W/G_W]) = H^\bullet\left(CH^\bullet([M_W/G_W]), \mathrm{d}_{CH}\right).$$

It is clear that $CH^\bullet([M_W/G_W])$ again encode an orbifold information. We get a direct sum decomposition

$$CW^\bullet([M_W/G_W]) = \bigoplus_{h\in G_W} CW^\bullet([M_W/G_W])^h,$$

where $CW^\bullet([M_W/G_W])^h$ consists of $(h,\gamma) \in \mathscr{L}([M_W/G_W])$. Differential and product respects this decomposition. A lift of a smooth pseudo-holomorphic cylinder $u \in \mathscr{M}_{1,1;0}(\gamma_+,\gamma_-)$ is a strip

$$\widetilde{u} : Z \to M_W, \tag{6.4.1}$$

$$u(\pm\infty, t) = \gamma_\pm(t), \tag{6.4.2}$$

$$g\cdot u(s,0) = u(s,1), \quad \forall s \in (-\infty,\infty). \tag{6.4.3}$$

rather than a cylinder. Therefore $\gamma_\pm$ must lie in a same direct summand. Similar result hold for a product structure. Namely, it restricts to

$$CH^\bullet([M_W/G_W])^{h_1} \otimes CH^\bullet([M_W/G_W])^{h_2} \to CH^\bullet([M_W/G_W])^{h_1h_2}.$$

In particular, the product structure commutative only because G_W is abelian.

We define a closed-open map

$$\mathrm{CO} : CH^\bullet(M_W/G_W) \to CC^\bullet\left(\mathscr{W}\mathscr{F}([M_W/G_W]), \mathscr{W}\mathscr{F}([M_W/G_W])\right)$$

popsicle operation

$$m^\Gamma_{n,F,\phi}$$

and a new category

$$\mathscr{C}_\Gamma$$

associated to $\Gamma \in SH^\bullet([M_W/G_W])$ in a same manner as before. By the same reason as in 6.4.2, CO is a chain map. Also, $\mathscr{C}_\Gamma$ is an A_∞ category.

6.5 Floer algebra of Seidel's immersed Lagrangian $\mathbb{L}$ and its deformation

Since $[M_W/G_W]$ is an orbifold sphere with three special points, we can consider an immersed circle, called Seidel Lagrangian $\mathbb{L}$ and its A_∞-algebra following Seidel. (See Figure 6.1 and [Sei11]) We breifly recall the algebra structure of $CF^\bullet(\mathbb{L},\mathbb{L})$. It has immersed generators X, Y, Z of odd degree, $\bar{X} = Y \wedge Z, \bar{Y} = Z \wedge X, \bar{Z} = X \wedge Y$ of even degree.

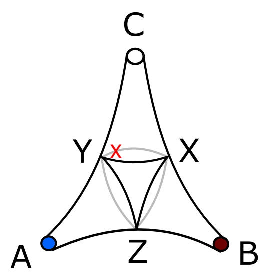


Figure 6.1: Orbifold sphere $\mathbb{P}^1_{a,b,c}$ in the Fermat case with one puncture C

Its Ω and H^1-grading is given by the following table.

	$1_{\mathbb{L}}$	X	Y	Z	$\bar{X}$	$\bar{Y}$	$\bar{Z}$	$[pt] = X \wedge Y \wedge Z$
Ω-grading	0	1	1	-1	2	0	0	1
H^1-grading	0	$-\gamma_2$	$-\gamma_1$	$-\gamma_3$	γ_2	γ_1	γ_3	0

Using the reflection symmetry (take $\mathbb{L}$ to be invariant under the reflection), we can follow [CHL17] to prove that $\mathbb{L}$ is weakly unobstructed and comput its potential function, denoted as $\widetilde{W}$.

Lemma 6.5.1. *Equip $\mathbb{L}$ with a nontrivial spin structure (marked as red crossing in Figure 6.1). Then*

1. *$\mathbb{L}$ is weakly unobstructued.*

2. *$b = xX + yY + zZ$ is a weak bounding cochains with potential $\widetilde{W}$ where*

$$\widetilde{W} = \begin{cases} x^p + y^q + xyz, & \textit{for } F_{p,q} \\ x^q + xyz, & \textit{for } C_{p,q} \\ xyz, & \textit{for } L_{p,q} \end{cases}$$

Remark 6.5.2. *Note that $\widetilde{W}$ is independent of p for the chain, p, q for the loop case. This is not a contradiction because the quotient space $[M_W/G_W]$ is also independent of those indices as well*

Remark 6.5.3. *Since Milnor fibers are exact and $\mathbb{L}$ is exact Lagrangian (since it is homologically trivial), there exist a change of coordinate such that W does not have any area T-coefficient. Therefore, we will omit them in the paper.*

$\mathbb{L}$ is oriented so that edges of the front xyz triangle are oriented counter-clockwise.

Proof. Weakly unobstructedness can be proved exactly the same way as Theorem 7.5 of [CHL17]. To compute $\widetilde{W}$, we fix a generic point and count all polygons whose corners are given by X, Y, Z's. Because of punctures, there are finitely many polygons contributing to $\widetilde{W}$. Also, we are only counting smooth discs, which have lifts to the Milnor fiber. So we can count them in the cover.

In the Fermat case, recall that the Milnor fiber can be obtained by first taking $2p$-gon and taking $2q$-copies of these $2p$-gon's by rotation around the $\mathbb{Z}/2q$ fixed point. Then, we have one $2p$-gon and one $2q$-gon and XYZ-triangle passing through a generic point. See Figure.. Therefore, we have

$$\widetilde{W} = x^p + y^q + xyz$$

In the Chain case, its Milnor fiber is given by a $2pq$-gon with A-puncture at the center and B-vertex and C-puncture as vertices of $2pq$-gon (with $\mathbb{Z}/pq$-action around A). To see the discs, it is more convenient to cut this into pq-pieces along rays connecting A and C. We can glue q of them around the B-vertex ($\mathbb{Z}/q$-fixed point) to obtain $2q$-gon, and as there are p many B-vertices, we get p many $2q$-gons. (see Figure ...) Each $2q$-gon has all the vertices as punctures, and one can check that a rigid holomorphic polygon with boundary on $\mathbb{L}$ has to be contained in one of the $2q$-gon. By inspection we obtain

$$\widetilde{W} = y^q + xyz$$

In the Loop case, all vertices are punctures and one can easily check that the only nontrivial disc is XYZ-triangle. Hence we have

$$\widetilde{W} = xyz$$

□

For weakly unobstructed $\mathbb{L}$, localized mirror functor formalism ([CHL17]) provides a canonical A_∞-functor from Fukaya category of $[M_W/G_W]$ to the matrix factorization category of $\widetilde{W}$.

We recall its definition from [CHL17] to set the notations.

6.6 Localized mirror functor to Matrix factorization category

Let R be a polynomial algebra over the algebraically closed field k of characteristic 0.

Definition 6.6.1. *For $f \in R$ a matrix factorization of f is defined by a pair (P, d) where P is $\mathbb{Z}/2\mathbb{Z}$-graded free R-module and d is an odd degree endomorphism such that $d^2 = f \cdot id$. The dg-category of matrix factorizations $MF_{dg}(f)$ of f is defined as follows. An object of $MF_{dg}(f)$ is a matrix factorization, and $\hom_{MF_{dg}(f)}(P, P')$ is given by $\mathbb{Z}/2\mathbb{Z}$-graded R-module maps between P and P', with usual composition $\circ$. A differential d on homogeneous morphisms are defined by*

$$d(\phi) = d_{P'} \circ \phi - (-1)^{deg(\phi)} \phi \circ d_P.$$

It is more convenient to use A_∞-category $\mathcal{MF}(f)$ for mirror symmetry.

Definition 6.6.2. *An A_∞-category $\mathcal{MF}(f)$ is defined as follows. An object of $\mathcal{MF}(f)$ is a matrix factorization of f. For two matrix factorizations P and P', its morphism space is*

$$\hom_{\mathcal{MF}(f)}(P, P') = \hom_{MF_{dg}(f)}(P', P)$$

with A_∞-operations

$$m_1 := d, \; m_2(\phi, \psi) := (-1)^{deg(\phi)} \phi \circ \psi, \; m_{\geq 3} = 0.$$

Definition 6.6.3. *Let $W^{\mathbb{L}}$ be the disk potential of $\mathbb{L}$. The localized mirror functor $\mathscr{F}^{\mathbb{L}} : Fuk(X) \to MF_{A_\infty}(W^{\mathbb{L}})$ is defined as follows.*

- *For given Lagrangian L, $\mathscr{F}^{\mathbb{L}}(L) := (CF(L, \mathbb{L}), -m_1^{0,b}) =: M_L$.*
- *Higher component*

  $$\mathscr{F}^{\mathbb{L}}_k : CF(L_1, L_2) \otimes \cdots \otimes CF(L_k, L_{k+1}) \to MF_{A_\infty}(M_{L_1}, M_{L_{k+1}})$$

  is given by

  $$\mathscr{F}^{\mathbb{L}}_k(a_1, \cdots, a_k) := \sum_{i=0}^{\infty} m_{k+1+i}(a_1, \cdots, a_k, \bullet, \overbrace{b, \cdots, b}^{i}).$$

  Here the input $\bullet$ is an element in $M_{L_{k+1}} = CF(L_{k+1}, \mathbb{L})$.

This construction based on Lagrangian Floer theory gives an A_∞-functor.

Theorem 6.6.4 (Theorem 2.19 [CHL17])**.** *$\mathscr{F}^{\mathbb{L}}$ is a covariant A_∞-functor.*

Chapter 7

Homological mirror symmetry for Milnor fibers of invertible curve singularity

In this section, we consider homological mirror symmetry for Milnor fiber as a symplectic manifold. We will find that G_W-equivariant mirror of M_W is a Landau-Ginzurg model $\widetilde{W}$. By applying Theorem 6.6.4 to the wrapped Fukaya category of $[M_W/G_W]$, we obtain an A_∞-functor, which is shown to give derived equivalence.

Theorem 7.0.1. *We have an A_∞-functor $\mathscr{F}^{\mathbb{L}}$*

$$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_W/G_W]) \to \mathscr{M}\mathscr{F}(\widetilde{W})$$

where $\widetilde{W}$ for Fermat $F_{p,q}$, Chain $C_{p,q}$ and loop $L_{p,q}$ cases are given as

$$\widetilde{W} = x^p + y^q + xyz, \quad x^q + xyz, \quad xyz$$

This functor is fully faithful and gives a derived equivalence between two categories.

Remark 7.0.2. *$\widetilde{W}$ is related to the transposed potential W^T as follows. If we set*

$$g(x, y, z) = z, z - y^p, z - x^{p-1} - y^{q-1}$$

then we have

$$\widetilde{W} = W^T(x, y) + xyg.$$

As we will explain later, if we add monodromy information and take our newly defined A_∞-category, the mirror will be obtained by setting $g = 0$, hence we obtain the matrix factorization of $W^T(x, y)$.

We prove the above theorem in the rest of the section. Although we treat each cases separately, the underlying strategies are basically the same.

7.1 Fermat cases

Recall that $G_W = \mathbb{Z}/p \times \mathbb{Z}/q$ is the maximal diagonal symmetry of $W = x^p + y^q$ and the quotient space $[M_{F_{p,q}}/G_W]$ has a single puncture say C of orbifold order $\frac{pq}{gcd(p,q)}$. Then, for a preimage $\widetilde{C}$ in $M_{F_{p,q}}$, we connect $\widetilde{C}$ and $(1,0)\cdot\widetilde{C}$ by a shortest path $\widetilde{L}_1$ as in the Figure ..., which we take as a non-compact Lagrangian. We denote by L the embedded Lagrangian in $[M_{F_{p,q}}/G_W]$ given by its projection. Denote by $\widetilde{L}$ the set of lifts of L in $M_{F_{p,q}}$, which is exactly $G_W \cdot \widetilde{L}_1$.

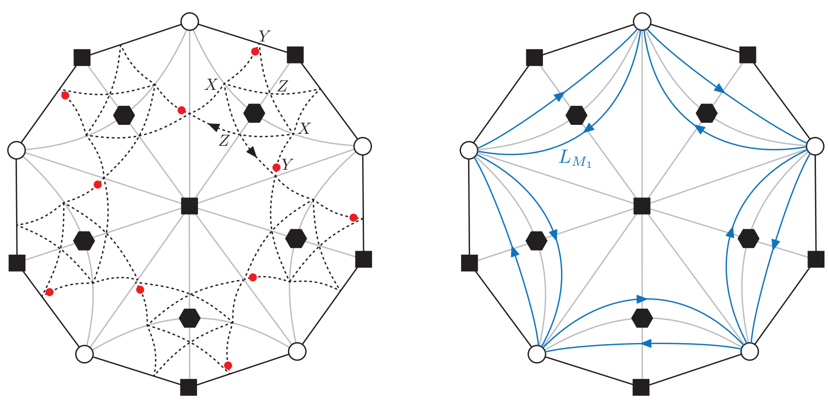


Figure 7.1: Milnor fiber of $F_{4,2}$ and a choice of Lagrangian L and its lifts $\widetilde{L}$

To prove the theorem in Fermat case, we show that G_W copies of L split-generates $\mathscr{W}\mathscr{F}(M_{F_{p,q}})$. Also, we compute the mirror matrix factorization $\mathscr{F}^{\mathbb{L}}(L)$, and show that the functor is fully faithful. Finally, we show that $\mathscr{M}\mathscr{F}(\widetilde{W})$ is split generated by $\mathscr{F}^{\mathbb{L}}(L)$, and this proves the theorem 7.0.1.

Wrapped Fukaya category of a surface is rather well-known Since G_W acts freely on objects, it is easy to see that wrapped Fukaya category of $M_{F_{p,q}}$ has a strict G_W-action, and we have

$$CW^{\bullet}(L,L) = CW^{G_W,\bullet}(\widetilde{L},\widetilde{L})$$

Lemma 7.1.1. *Wrapped Floer complex $CW^{\bullet}(L,L)$ is quasi-isomorphic to the following model.*

1. *As a vector space,*
$$CW^{\bullet}(L,L) \simeq T(a,b)/\mathscr{R}_{F_{p,q}}$$
Here, $T(a,b)$ is a tensor algebra generates by two alphabets a,b. The ideal $\mathscr{R}_{F_{p,q}}$ is defined as
$$\mathscr{R}_{F_{p,q}} =< a\otimes a = \delta_{2,p}, b\otimes b = \delta_{2,q} >$$
$\mathbb{Z}/2$-grading of a,b is odd and this induces $\mathbb{Z}/2$-grading on $T(a,b)/\mathscr{R}_{F_{p,q}}$.
2. *m_1 vanishes and m_2 coincides with the tensor product.*
3. *$m_k(a,\dots,a)$ is zero for $1 \le k < p$ and it is equal to 1 for $k = p$. Likewise, $m_k(b,\dots,b)$ is zero for $1 \le k < q$ and equal to 1 for $k = q$*

Its Ω and H^1-grading is given by the following table.

	1_L	a	b
Ω-*grading*	0	1	1
H^1-*grading*	0	$-\gamma_3$	$-\gamma_3$

Proof. We can choose $\widetilde{L}_1$ so that G_W-orbits of $\widetilde{L}_1$ are disjoint. Therefore $CW^{\bullet}(L,L)$ consists only of hamiltonian chords at an infinity. Among such chords we choose the following two generators.

- a, the shortest chord $\in CW^{\bullet}(\widetilde{L}_1,(1,0)\cdot\widetilde{L}_1)$, $(1,0) \in \mathbb{Z}/p\times\mathbb{Z}/q$
- b, the shortest chord $\in CW^{\bullet}(\widetilde{L}_1,(0,1)\cdot\widetilde{L}_1)$, $(0,1) \in \mathbb{Z}/p\times\mathbb{Z}/q$.

For example, take a rotation of $\widetilde{L}_1$ around the $\mathbb{Z}/p$ fixed point and there is a unique wrapped generator a between these two branches. By abuse of notation, we also denote by a, b the generators in $CW^{G_W,\bullet}(\widetilde{L},\widetilde{L})$ given by the sum of G_W-copies of the above generators.

We can also concatenate them to create new hamiltonian chords (m_2-products near the puncture), denoted by $\{a, b, ab, ba, aba, bab, \ldots\}$. One can check that m_1 vanishes. Note that if we consider m_2-operations near the puncture, $m_2(a,a)$, $m_2(b,b)$ vanishes as they are not composable. If $p = 2$ or $q = 2$, we could have an m_2-product coming from a global holomorphic polygon which contributes $m_2(a,a)$ or $m_2(b,b)$ respectively. In general, there are two global J-holomorphic polygons with all of its corners are of word length 1. They are p-gon and q-gon and come from lifts of upper/lower hemisphere of $(M_{F_{p,q}}/G_W)\setminus L)$. Their corners are hamiltonian chords a or b at infinity. They cannot contribute to m_{p-1} or m_{q-1}, only contribute to m_p or m_q respectively. The boundaries of these polygons are whole G_W-orbits of L so they represents the unit element of $CW^{G_W,\bullet}(L,L)$. $\square$

Lemma 7.1.2. *$\widetilde{L}$ split-generate the wrapped Fukaya category of $M_{F_{p,q}}$*

Proof. We proceed as in the work of Heather Lee [Lee16]. To avoid confusion, let us denote by $\tilde{a}$ the sum over G_W orbit of a in this proof. From Abouzaid's generating criterion, it is enough to show that the following open-closed map hits the unit.

$$\mathcal{OC} : CC_\bullet(CW^\bullet(\widetilde{L},\widetilde{L})) \to SH^\bullet(M_{F_{p,q}})$$

We take the following Hochschild cycle

$$\frac{\tilde{a}^{\otimes p}}{p} - \frac{\tilde{b}^{\otimes q}}{q} \in CC_\bullet(CW^\bullet(\widetilde{L},\widetilde{L}))$$

It is not hard to see that $\widetilde{L}$ provides a tessellation of $M_{F_{p,q}}$, which consists of q distinct p-gons and p distinct q-gons. We first check that it is G_W-equivariant Hochschild cycle. From Lemma 7.1.1, it is enough to check m_p, m_q operations respectively.

$$\partial_{Hoch}(\tilde{a}^{\otimes p}/p - \tilde{b}^{\otimes q}/q) = (m_p(\tilde{a},\ldots,\tilde{a}) - m_q(\tilde{b},\ldots,\tilde{b})) = 1_{\widetilde{L}} - 1_{\widetilde{L}} = 0.$$

On the other hand, the image of the open-closed map of this Hochschild cycle is a cocycle represented by the bounded area of $M_{F_{p,q}}$ covering each region with weight one. Note that the orientation of the boundary Lagrangian of p-gon and q-gon are opposite to each other, and thus p-gons and q-gons in the image add up despite the negative sign in the expression $-\tilde{b}^{\otimes q}/q$. □

Let us discuss the mirror matrix factorization. Using localized mirror functor, we can explicitly compute the mirror matrix factorization. Since $\widetilde{W}$ has non-isolated singularity (singularity along z-axis), we need to be a little bit careful in the discussion. Denote by $S = \mathbb{C}[x,y,z]$. By counting appropriate polygons from the picture with sign, we can prove the following lemma, whose proof is left as an exercise.

Lemma 7.1.3. *The localized mirror functor*

$$\mathcal{F}^{\mathbb{L}} : \mathcal{WF}([M_{F_{p,q}}/G_{F_{p,q}}]) \to \mathcal{MF}(\widetilde{W})$$

sends L to the following matrix factorization

$$M_L = \left(S^{\oplus 2} \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} S^{\oplus 2} \right) \tag{7.1.1}$$

$$\delta_0 = \begin{pmatrix} x & y \\ -y^{p-1} & x^{q-1} + yz \end{pmatrix} \tag{7.1.2}$$

$$\delta_1 = \begin{pmatrix} x^{q-1} + yz & -y \\ y^{p-1} & x \end{pmatrix} \tag{7.1.3}$$

Remark 7.1.4. *If we set $z = 0$, this matrix factorization become a compact generator of $\mathcal{MF}(W^T)$ corresponding to skyscraper sheaf at the singular point.*

Corollary 7.1.5. *The matrix factorization M_L is of Koszul type. Namely we have an isomorphism*

$$M_L \cong \left(S[\theta_x, \theta_y], \partial_K + \partial'_K\right).$$

Here, θ_1, θ_2 are odd degree generators (hence anti-commute) and

$$\partial_K = x \cdot \iota_{\theta_x} + y \cdot \iota_{\theta_y}, \;\; \partial'_K := W_x \theta_x \wedge \cdot + W_y \theta_y \wedge \cdot$$

where

$$W_x = (x^{q-1} + yz), \quad W_y = y^{p-1}$$

Remark 7.1.6. *The following Koszul complex has cohomology* $\mathbb{C}[z]$ *concentrated on the right end.*

$$K(x,y) := 0 \to S(\theta_1 \wedge \theta_2) \xrightarrow{\partial_K} S\theta_1 \oplus S\theta_2 \xrightarrow{\partial_K} S \to 0$$

Therefore, following Dyckerhoff [Dyc11], we compute its endomorphism algebra $End(M_L)$.

Lemma 7.1.7. $End_{\mathcal{MF}_{dg}}(M_L)$ *is quasi-isomorphic to a DG algebra of polynomial differential operators*

$$\mathrm{Hom}_{\mathcal{MF}_{dg}}(M_L, M_L) \simeq \Big(S[\partial_{\theta_x}, \partial_{\theta_y}, (\theta_x \wedge), (\theta_y \wedge)], D \Big) \tag{7.1.4}$$

$$D(\partial_{\theta_x}) = W_x, \quad D(\partial_{\theta_y}) = W_y \tag{7.1.5}$$

$$D(\theta_x \wedge) = x, \quad D(\theta_y \wedge) = y \tag{7.1.6}$$

Its cohomology is

$$H^\bullet(\mathrm{Hom}_{\mathcal{MF}_{dg}}(M_L, M_L)) \simeq \mathbb{C}[z][\Gamma_x, \Gamma_y] \tag{7.1.7}$$

$$\Gamma_x = [\partial_{\theta_x} - x^{q-2}(\theta_x \wedge) - z(\theta_y \wedge)] \tag{7.1.8}$$

$$\Gamma_y = [\partial_{\theta_y} - y^{p-2}(\theta_y \wedge)] \tag{7.1.9}$$

Proof. The first part of the lemma is obvious because morphisms of a matrix factorization of Koszul type are those of exterior algebras. It is easy to check that the differential satisfies given equations. For example,

$$D(\partial_{\theta_x}) = [\partial_K + \partial'_K, \partial_{\theta_x}] \tag{7.1.10}$$

$$= [\partial'_K, \partial_{\theta_x}] \tag{7.1.11}$$

$$= [(W_x \theta_x \wedge), \iota_{\theta_x}] = W_x \tag{7.1.12}$$

To each differential operators we can assign an order of its symbols. It provides a decreasing filteration $\{F^i\}$ on the complex. The first page of the spectral sequence associated to the filtration is a dual Koszul complex associated to a regular sequence (x, y)

$$E_1 = H^\bullet\Big(K^\vee(x,y)\otimes_{\mathbb{C}}\mathbb{C}[\partial_{\theta_x},\partial_{\theta_y}],\partial_K^\vee\otimes 1\Big)\simeq \mathbb{C}[z][\partial_{\theta_x},\partial_{\theta_y}]$$

In particular we know that the cohomology algebra is a $\mathbb{C}[z]$- modules of rank less or equal to 4. On the other hand, the cycles generated by Γ_x, Γ_y in the lemma have already provided four $C[z]$-linear independent element. Therefore the spectral sequence degenerates at E_1 page. This finishes the proof. □

We can show that our mirror functor is fully-faithful.

Lemma 7.1.8. *The first-order part of the mirror functor is*

$$\mathscr{F}_1^{\mathbb{L}} : CW^\bullet(L,L) \to Hom_{\mathscr{MF}}(M_L, M_L) \tag{7.1.13}$$

$$a \to \Gamma_x \tag{7.1.14}$$

$$b \to \Gamma_y. \tag{7.1.15}$$

It is a quasi-isomorphism. Therefore $\mathscr{F}^{\mathbb{L}}$ embedds $\mathscr{W}\mathscr{F}(M_{F_{p,q}})$ as a full subcategory of $\mathscr{MF}(\widetilde{W})$.

Proof. From the Figure 7.1, we see that $\mathscr{F}_1^{\mathbb{L}}$ sends a to Γ_x and b to Γ_y. Moreover,

$$[\Gamma_x, \Gamma_y] = [-z(\theta_y\wedge), \partial_{\theta_y}] = z.$$

Therefore $ab + ba$ hits z and $\mathscr{F}_1^{\mathbb{L}}$ is surjective.

Notice that $CW^\bullet(L,L)$ and $H^\bullet(Hom_{\mathscr{MF}}(M_L, M_L))$ are filtered by

$$F^k := (ab+ba)^k \cdot CW^\bullet(L,L), \quad G^l := z^l \cdot H^\bullet(Hom_{\mathscr{MF}}(M_L, M_L))$$

It is easy to check that $\mathscr{F}_1^{\mathbb{L}}$ is a filtered map with respect to $F^\bullet$ and $G^\bullet$.

The graded piece F^0/F^1 is a 4 dimensional vector space spanned by four words $< 1, a, b, ab >$. This is because

$$aba = (ab+ba)\cdot a - \delta_{2,q} b, \quad bab = (ab+ba)\cdot b - \delta_{2,p} a.$$

An element $ab + ba$ is in the center of the algebra. Therefore

$$F^k/F^{k+1} \simeq (ab+ba)^k \cdot F^0/F^1 = (ab+ba)^k \cdot < 1, a, b, ab > .$$

By a similar reason, we have

$$G^k/G^{k+1} \simeq z^k \cdot G^0/G^1 = z^k \cdot <1, \Gamma_x, \Gamma_y, (\Gamma_x \circ \Gamma_y)>$$

The induced morphism of associated graded $Gr\mathscr{F}_1^{\mathbb{L}}$ is an isomorphism of vector spaces at every level. By the comparison theorem, so is $\mathscr{F}_1^{\mathbb{L}}$. □

Corollary 7.1.9. *$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_{F_{p,q}}/G_{F_{p,q}}]) \to \mathscr{M}\mathscr{F}(W^T + xyz)$ is a quasi-equivalence.*

Proof. It is enough to show that M_L and $M_{\mathbb{L}}$ generates MF$(W^T + xyz)$. Orlov's equivalence

$$\mathrm{MF}(W^T + xyz) \simeq D_{sg}(W^T + xyz)$$

$$\left(M^1 \underset{\psi}{\overset{\phi}{\rightleftarrows}} M^0 \right) \mapsto Coker(\psi)$$

sends $M_{\mathbb{L}}$ to a skyscraper sheaf $\mathscr{O}_o$ at the orgin and M_L to a structure sheaf $\mathscr{O}_z$ of z-axis. These are two irreducible components of a critical locus of $W^T + xyz$. Therefore it generates MF$(W^T + xyz)$.(See [Ste13]) □

7.2 Chain cases

The polynomial $W = C_{p,q} = x^p + xy^q$ has maximal symmetry group $G_W = \mathbb{Z}/pq$. We proceed as in the Fermat case. Denote by ξ the following generator of G_W:

$$x \to e^{\frac{2\pi i}{p}} \cdot x, \quad y \to e^{\frac{-2\pi i}{pq}} \cdot y.$$

Recall that the quotient space $[M_{C_{p,q}}/G_W]$ has one orbifold points of order q and two punctures of order pq and $\frac{pq}{gcd(p-1,q)}$, respectively. Let us call them as B_1, B_2 respectively. The orbifold action near B_1 is generated by ξ while the action near B_2 is generated by ξ^{p-1} by Proposition 5.2.1.

We take a Lagrangian L connecting B_1 and B_2 in $\mathbb{P}^1_{pq,q,\frac{pq}{gcd(p-1,q)}}$ (we may take the part of the equator between B_1 and B_2). And denote by $\widetilde{L}$ the sum of all lifts of L in the Milnor fiber.

Lemma 7.2.1. *The wrapped Floer complex $CW^\bullet(L,L)$ is quasi-isomorphic to the following model.*

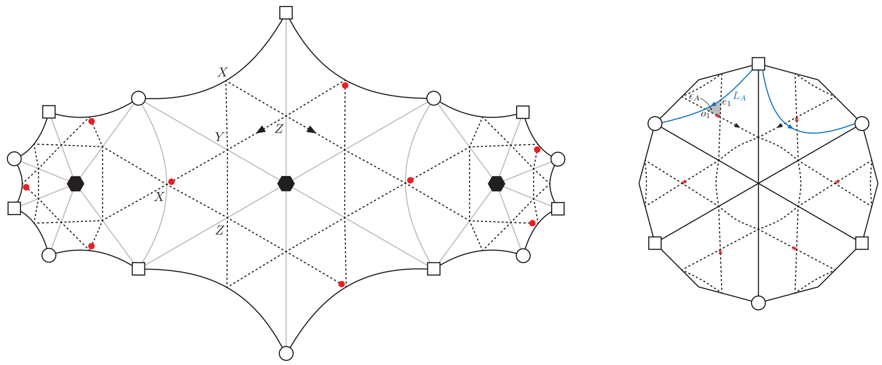


Figure 7.2: Milnor fiber of $E7 = C_{3,3}$ and a choice of Lagrangian L

1. *As a vector space,*
$$CW^\bullet(L,L) \simeq \mathbb{C}[a,b]/(ab=0)$$
Here, a, b are even variables.

2. *m_1 vanishes and m_2 coincides with a polynomial multiplication.*

3. *$m_k(a,b,a,b\ldots) = 0$ for $1 \le k \le 2q-1$ and $m_{2q}(a,b,\ldots,a,b) = 1$. Likewise, $m_k(b,a,b,a\ldots) = 0$ for $1 \le k \le 2q-1$ and $m_{2q}(b,a,\ldots,b,a) = 1$*

Its Ω and H^1-grading is given by the following table.

	1_L	a	b
Ω-grading	0	0	2
H^1-grading	0	$-\gamma_1$	$-\gamma_3$

Proof. Branches of $\widetilde{L}$ don't intersect with each other in the interior. Therefore $CW^\bullet(L,L)$ consists of hamiltonian chords at infinity near B_1 or B_2. Among them we choose two generators between the nearest orbits. Namely, choose one lift $\widetilde{L}_1$ and take the wrapped generator

- a, the shortest chord $\in CW^\bullet(\widetilde{L}_1, \xi^{-1}\cdot\widetilde{L}_1)$ near B_1
- b, the shortest chord $\in CW^\bullet(\widetilde{L}_1, \xi^{1-p}\cdot\widetilde{L}_1)$ near B_2.

Here, a (resp. b) is nothing but the chord between $\widetilde{L}_1$ and its clockwise rotation at B_1 (resp. B_2). Namely, recall that ξ, ξ^{p-1} correspond to γ_1, γ_3 of the orbifold fundamental group in the Proposition 5.2.1. And $\gamma_1^{-1}, \gamma_3^{-1}$ are the minimal clockwise rotations in the uniformizing neighborhood of orbifold points. Therefore $\xi \cdot \widetilde{L}_1$ is obtained by clockwise rotation of $\widetilde{L}_1$(centered at B_1) sending B_2-vertex to the nearest B_2-vertex. The same holds for $\xi^{p-1} \cdot \widetilde{L}_1$ switching the role of B_1 and B_2.

We can also concatenate them to create new Hamitonian chords, namely $a^2, a^3, \ldots, b^2, b^3, \ldots$. We cannot concatenate different words as their heads and tails are different from each other. The rest of the argument is similar to the Fermat case. m_1 vanishes because there are no J-holomorphic strip between them. Concatenating two chords corresponds to m_2 operation concentrated near the punctures. The first global J-holomorphic polygon contributes to a non-trivial A_∞ operation is a $2q$-gon. It is a lift of an orbifold bigon $(M_{C_{p,q}}/G_W) \setminus L$. It's corners consists of q many a and b alternating each other. □

Lemma 7.2.2. *$\widetilde{L}$ generates the wrapped Fukaya category of $M_{C_{p,q}}$*

Proof. We proceed as in the Fermat case. Milnor fiber $M_{C_{p,q}}$ is tessellated by p copies of $2q$-gons that are considered in the previous lemma. In Figure 7.2, this is given by 3 copies of hexagons. To show that open-closed map hits the unit, we take the following Hochschild cycle.

$$\frac{1}{q}(\tilde{a} \otimes \tilde{b})^{\otimes q} \in CC_\bullet(CW^\bullet(\widetilde{L}, \widetilde{L}))$$

It is indeed a cycle because

$$\partial_{Hoch}\left(\frac{1}{q}(\tilde{a} \otimes \tilde{b})^{\otimes q}\right) = m_{2q}(\tilde{a}, \tilde{b}, \ldots, \tilde{a}, \tilde{b}) - m_{2q}(\tilde{b}, \tilde{a}, \ldots, \tilde{b}, \tilde{a}) \tag{7.2.1}$$

$$= 1_{\widetilde{L}} - 1_{\widetilde{L}} = 0 \tag{7.2.2}$$

On the other hand, the image of the open-closed map of this Hochschild cycle is a cocycle represented by the bounded area of $M_{C_{p,q}}$ covering each region with weight one. □

If we solve the weak Maurer-Cartan equation for L, we get the potential $\widetilde{W} = x^q + xyz$, which can be also written as $W^T + xyg$ with $g(x, y, z) = z - y^{p-1}$.

Lemma 7.2.3. *The localized mirror functor*

$$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_{C_{p,q}}/G_{C_{p,q}}]) \to \mathscr{M}\mathscr{F}(\widetilde{W})$$

sends L to the following matrix factorization

$$M_L = \Big(\; S \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} S \; \Big) \tag{7.2.3}$$

$$\delta_0 = x \tag{7.2.4}$$

$$\delta_1 = x^{q-1} + yz \tag{7.2.5}$$

Proof. This follows from the Figure 7.2. □

The matrix factorization M_L we get is again of Koszul type. It is even simpler; it is an actual factorization of $\tilde{W}$. One can check directly that

$$M_L = \big(S[\theta_x], (x \cdot i_{\theta_x} + W_x \cdot \theta_x \wedge)\big), \quad W_x = x^{q-1} + yz$$

Using the same technique,

Lemma 7.2.4. *The self-hom space of M_L is quasi-isomorphic to a DG algebra of polynomial differential operators*

$$Hom_{\mathrm{MF}}(M_L, M_L) \simeq \Big(S[\partial_{\theta_x}, (\theta_x \wedge)], D\Big) \tag{7.2.6}$$

$$D(\partial_{\theta_x}) = W_x, \tag{7.2.7}$$

$$D(\theta_x \wedge) = x. \tag{7.2.8}$$

Its cohomology is concentrated to even degree, isomorphic to

$$H^{\bullet}(Hom_{\mathrm{MF}}(M_L, M_L)) \simeq \mathbb{C}[y,z]/(yz = 0)$$

Proof. The first part of the lemma is same as Fermat case. The cohomology computation can be done in a similar way, but we found that it is much easier to do it by hands. This complex is isomorphic to the 2-periodic complex

$$S^{\oplus 2,even} \underset{D_1}{\overset{D_0}{\rightleftarrows}} S^{\oplus 2,odd} \tag{7.2.9}$$

$$D_0 = \begin{pmatrix} x & -x \\ W_x & -W_x \end{pmatrix} \tag{7.2.10}$$

$$D_1 = \begin{pmatrix} x & -W_x \\ x & -W_x \end{pmatrix} \tag{7.2.11}$$

Therefore, we have

$$Ker(D_0)/Im(D_1) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,even} \middle| a = b \right\} \Big/ S \cdot \begin{pmatrix} x \\ x \end{pmatrix} \oplus S \cdot \begin{pmatrix} W_x \\ W_x \end{pmatrix} \tag{7.2.12}$$

$$\simeq \mathbb{C}[x,y,z]/(x = x^{p-1} + yz = 0) \tag{7.2.13}$$

$$\simeq \mathbb{C}[y,z]/(yz = 0) \tag{7.2.14}$$

$$Ker(D_1)/Im(D_0) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,odd} \middle| W_x \cdot a + x \cdot b = 0 \right\} \Big/ S \cdot \begin{pmatrix} x \\ -W_x \end{pmatrix} \tag{7.2.15}$$

$$\simeq 0 \tag{7.2.16}$$

The last equality holds because (x, W_x) is a regular sequence of S. □

Now we can show that our mirror functor is an equivalence.

Lemma 7.2.5. *The first-order part of the mirror functor is given by*

$$\mathscr{F}_1^{\mathbb{L}} : CW^{\bullet}(L,L) \to Hom_{\mathscr{MF}}(M_L, M_L) \tag{7.2.17}$$

$$a \to y \tag{7.2.18}$$

$$b \to z. \tag{7.2.19}$$

It is a quasi-isomorphism. Moreover $\mathscr{F}^{\mathbb{L}} : \mathscr{WF}([M_{C_{p,q}}/G_{C_{p,q}}]) \to \mathscr{MF}(x^{q-1} + xyz)$ *is a quasi-equivalence.*

Proof. Similar to Fermat case. □

7.3 Loop cases

The loop type polynomial $W = x_1^p x_2 + x_1 x_2^q$ has $G_W = \mathbb{Z}/pq-1$ as the maximal diagonal symmetry group. One notable difference of a loop type from the others is that the action of the maximal diagonal symmetry is free. The quotient $M_{L_{p,q}}/G$ is an honest three punctured sphere. Its wrapped Fukaya category and its homological mirror symmetry was proved in [AAE+13]. The result in this section can be essentially found therein, except that we use localized mirror functor to define the explicit correspondences.

Let us introduce more notation. For loop type, we use variables x_i ($i = 1,2,3$) instead of x, y, z. Let ξ denote the following generators of this group.

$$x_1 \to e^{\frac{2q\pi i}{pq-1}} \cdot x_1, \quad x_2 \to e^{\frac{-2\pi i}{pq-1}} \cdot x_2$$

Also recall three puctures are of order $pq-1, pq-1$ and $\frac{pq-1}{gcd(p-1,q-1)}$. Let's denote them by B_1, B_2, B_3 respectively. A cyclic orbifold action is generated by ξ near B_1, by ξ^{-p} near B_2 and by ξ^{1-p} near B_3 by the Proposition 5.2.1. As there are three punctures, we choose three shortest Lagrangians L_i from B_{i+1} to B_{i+2} for $i = 1,2,3 \mod 3$ which are part of the equator sphere passing through 3 punctures. The following can be checked from [AAE+13].

Lemma 7.3.1. *The wrapped Floer complexes $CW^\bullet(L_i, L_j)$ is quasi-isomorphic to the following model.*

1. *as a vector space,*

$$CW^\bullet(L_i, L_j) \simeq \begin{cases} \mathbb{C}[a_{i+1}, b_{i+2}]/(a_{i+1}b_{i+2} = 0) & i = j \\ \mathbb{C} < a_i^n \cdot c_{i,j} \cdot b_j^m >, \ n, m \in \mathbb{N} & i \neq j \end{cases}$$

   Here, a_i, b_i are even an and $c_{i,j}$ are odd.

2. *m_1 vanishes and m_2 coincides with a polynomial multiplication and an obvious bimodule structure.*

3. $m_3(c_{12}, c_{23}, c_{31}) = 1$

Its Ω and H^1-grading is given by the following table.

	1_{L_i}	a_i	b_{i+1}	c_{12}	c_{23}	c_{31}
Ω-*grading*	0	$2\cdot\delta_{i,3}$	$2\cdot\delta_{i,3}$	-1	1	1
H^1-*grading*	0	$-\gamma_i$	$-\gamma_i$		*not defined*	

Consider the direct sum of lifts $\widetilde{L}_i$ of L_i in $M_{L_{p,q}}$. Then the following is well-known.

Lemma 7.3.2. $\{\widetilde{L}_i\}_{i=1,2,3}$ *split-generates the wrapped Fukaya category of* $M_{L_{p,q}}$

Next, we move on to the mirror computation. For the Seidel Lagrangian $\mathbb{L}$ in the quotient $[M_{L_{p,q}}/G_W]$, the potential function can be computed as

$$\widetilde{W} = xyz.$$

Remark 7.3.3. *We may write*

$$xyz = x^p y + xy^q + xy(z - x^{p-1} - y^{q-1}) = W^t + xy\cdot g(x,y,z)$$

From the picture, it is easy to check the following.

Lemma 7.3.4. *The localized mirror functor*

$$\mathscr{F}^{\mathbb{L}} : \mathscr{W}\mathscr{F}([M_{L_{p,q}}/G_{L_{p,q}}]) \to \mathscr{M}\mathscr{F}(x_1x_2x_3)$$

sends L to a following matrix factorization

$$M_{L_i} = (\ S \underset{\delta_{i,1}}{\overset{\delta_{i,0}}{\rightleftarrows}} S\ ) \tag{7.3.1}$$

$$\delta_0 = x_i,\ \ \delta_1 = \frac{x_1x_2x_3}{x_i} \tag{7.3.2}$$

As before, we can also write this matrix factorization as

$$M_{L_i} = \big(S[\theta_{x_i}], (x_i \cdot i_{\theta_{x_i}} + W_{x_i}\cdot\theta_{x_i}\wedge)\big), \quad W_{x_i} = \frac{x_1x_2x_3}{x_i}.$$

For later purpose, we calculate hom complex by hand. (in [AAE+13] it was proved using Orlov's equivalence.)

Lemma 7.3.5. *The self-hom space of M_{L_i} is quasi-isomorphic to a DG algebra of polynomial differential operators*

$$Hom_{\mathrm{MF}}(M_{L_i}, M_{L_i}) \simeq \left(S[\partial_{\theta_{x_i}}, (\theta_{x_i}\wedge)], D\right) \tag{7.3.3}$$

$$D(\partial_{\theta_x}) = W_{x_i}, \tag{7.3.4}$$

$$D(\theta_x \wedge) = x_i. \tag{7.3.5}$$

The cohomology of Floer complexes are given as follows;

$$H^\bullet(Hom_{\mathrm{MF}}(M_{L_i}, M_{L_i})) \simeq \begin{cases} \mathbb{C}[x_1, x_2, x_3]/(x_i = W_{x_i} = 0) & i = j \\ \mathbb{C}[x_1, x_2, x_3] \cdot (\frac{x_1 x_2 x_3}{x_i x_j})/(x_i = x_j = 0) & i \neq j \end{cases}$$

Proof. A computation of self-Floer complex is almost identical to that of chain type. A complex of morphism $Hom_{\mathrm{MF}}(M_{L_i}, M_{L_j})$ is isomorphic to

$$S^{\oplus 2, even} \underset{D_1}{\overset{D_0}{\rightleftarrows}} S^{\oplus 2, odd} \tag{7.3.6}$$

$$D_0 = \begin{pmatrix} x_i & -x_j \\ W_{x_j} & -W_{x_i} \end{pmatrix} \tag{7.3.7}$$

$$D_1 = \begin{pmatrix} W_{x_i} & -x_j \\ W_{x_j} & -x_i \end{pmatrix} \tag{7.3.8}$$

Therefore, we have

$$Ker(D_0)/Im(D_1) = \begin{cases} \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,even} \middle| a = b \right\} \Big/ S \cdot \begin{pmatrix} x_i \\ x_i \end{pmatrix} + S \cdot \begin{pmatrix} W_{x_i} \\ W_{x_i} \end{pmatrix} & (i = j) \\ \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2,even} \middle| x_i \cdot a = x_j \cdot b \right\} \Big/ S \cdot \begin{pmatrix} x_j \\ x_i \end{pmatrix} + S \cdot \begin{pmatrix} W_{x_i} \\ W_{x_j} \end{pmatrix} & (i \neq j) \end{cases} \tag{7.3.9}$$

$$\simeq \begin{cases} \mathbb{C}[x_1, x_2, x_3]/(x_i = W_{x_i} = 0) & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{7.3.10}$$

$$Ker(D_1)/Im(D_0) = \begin{cases} \left\{ \begin{pmatrix} c \\ d \end{pmatrix} \in S^{2,odd} \middle| W_{x_i} \cdot c = x_i \cdot d \right\} \Big/ S \cdot \begin{pmatrix} x_i \\ W_{x_i} \end{pmatrix} & (i = j) \\ \left\{ \begin{pmatrix} c \\ d \end{pmatrix} \in S^{2,odd} \middle| W_{x_i} \cdot c = x_j \cdot d \right\} \Big/ S \cdot \begin{pmatrix} x_i \\ W_{x_j} \end{pmatrix} + S \cdot \begin{pmatrix} x_j \\ W_{x_i} \end{pmatrix} & (i \neq j) \end{cases} \tag{7.3.11}$$

$$\simeq \begin{cases} 0 & (i = j) \\ \mathbb{C}[x_1, x_2, x_3] \cdot (\frac{x_1 x_2 x_3}{x_i x_j})/(x_i = x_j = 0) & (i \neq j) \end{cases} \tag{7.3.12}$$

□

Now we can show that our mirror functor is an equivalence as before.

Lemma 7.3.6. *The first-order part of the mirror functor is*

$$\mathscr{F}_1^{\mathbb{L}} : CW^{\bullet}(L_i, L_j) \to Hom_{\mathscr{MF}}(M_{L_i}, M_{L_j}) \tag{7.3.13}$$
$$a_i \to x_i \tag{7.3.14}$$
$$b_i \to x_i \tag{7.3.15}$$
$$c_{i,j} \to \frac{x_1 x_2 x_3}{x_i x_j} \tag{7.3.16}$$

It is a quasi-isomorphism. Moreover, $\mathscr{F}^{\mathbb{L}} : \mathscr{WF}([M_{C_{p,q}}/G_{L_{p,q}}]) \to \mathscr{MF}(x_1 x_2 x_3)$ *is a quasi-equivalence.*

Chapter 8

New Fukaya category for Landau-Ginzburg orbifolds

For a weighted homogeneous polynomial $W : \mathbb{C}^n \to \mathbb{C}$, let G_W be the maximal diagonal symmetry group which can be defined as in the two variable cases. Landau-Ginzburg orbifold is a pair (W, G') for a choice of subgroup $G' < G_W$. We plan to define a $\mathbb{Z}/2$-graded A_∞-category for (W, G').

8.1 Preliminaries

Definition 8.1.1. *A polynomial W is called weighted homogeneous polynomial if*

$$W(\lambda^{w_1} z_1, \cdots, \lambda^{w_n} z_n) = \lambda^h W(z_1, \cdots, z_n)$$

for $w_1, \cdots, w_n, h \in \mathbb{Z}$. We say W has weight $(w_1, \cdots, w_n; h)$. We will always assume that W has an isolated singularity at the origin. Namely, $\mathrm{grad} W = (\frac{\partial W}{\partial z_1}, \cdots, \frac{\partial W}{\partial z_n})$ *vanishes only at* $0 \in \mathbb{C}^n$.

We set $V_t = V_t(W) = \{z \in \mathbb{C}^n \mid W(z) = t\}$, and V_0 is an hypersurface of isolated singularity at 0 and V_t $(t \neq 0)$ is non-singular. Milnor fiber M_W is nothing but $V_1(W)$. For the well-known Milnor fibration

$$\frac{W}{|W|} : S_\epsilon^{2n-1} \setminus K \to S^1$$

with $K = (S^{2n-1}_\epsilon \cap V_0)$, its fiber is diffeomorphic to M_W. Geometric monodromy $h : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$h(x_1, \cdots, x_m) = (e^{2\pi i w_1/h} x_1, \cdots, e^{2\pi i w_m/h} x_m) \tag{8.1.1}$$

which restricts to $h : M_W \to M_W$. It is known that $S^{2n-1}_\epsilon \setminus K$ is diffeomorphic to the manifold obtained by identifying two ends of $M_W \times [0,1]$ by h. (see [Mil68] Lemma 9.4)

One can define its closure $\overline{M}_W$ and its boundary $\partial\overline{M}_W$. There are monodromy homomorphism (from a parallel transport fixing the boundary)

$$h_* : H_*(\overline{M}_W) \to H_*(\overline{M}_W), h_* : H_*(\overline{M}_W, \partial\overline{M}_W) \to H_*(\overline{M}_W, \partial\overline{M}_W) \tag{8.1.2}$$

A topological precursor of our construction is a **variation operator** (around the origin in $\mathbb{C}$)

$$\mathrm{var} : H_{n-1}(\overline{M}_W, \partial\overline{M}_W) \to H_*(\overline{M}_W). \tag{8.1.3}$$

It is defined by sending $[c] \to (h_* - id)([c])$.

We want to find a symplectic categorical analogue of this variation operator for weighted homogenous polynomials. At first, we will define a distinguished Reeb orbit Γ_W from the geometric monodromy (8.1.1). The analogue of monodromy homomorphism (8.1.2) will be the quantum cap action by Γ_W.

$$\cap\Gamma_W : \mathscr{W}\mathscr{F}(L, L) \to \mathscr{W}\mathscr{F}(L, L).$$

Then, the analogue of the variation operator (8.1.3) will be an assignment

$$L \mapsto \mathrm{Cone}\left(L \xrightarrow{\cap\Gamma_W} L \right)$$

Recall that in Floer theory, it is well-known that taking a Lagrangian surgery corresponds to taking a cone complex. Roughly speaking, we are taking a surgery of non-compact Lagrangians for the Reeb chords at infinity to turn it into a compact object, namely the corresponding vanishing cycle.

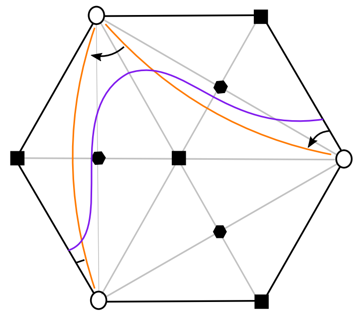


Figure 8.1: Lagrangian L, monodromy action and vanishing cycle

8.2 Monodromy, Reeb orbit, and $\mathscr{C}_{\Gamma_W}$

Let W be a n-variable weighted homogeneous polynomial of weight $(w_1,\dots,w_n;h)$. Consider a quadratic hamltonian

$$H := \frac{1}{2h}\sum_{1}^{n} w_i \cdot |x_i|^2$$

which generates a circle action of $\mathbb{C}^n$ of a given weight. The Hamiltonian flow $\Phi_W(s)$ of X_H is called *monodromy flow*.

Definition 8.2.1. *A monodromy transformation* $\Phi_W = \Phi_W(1)$ *is a time-*1 *hamiltonian flow of* H.

$$x_i \mapsto e^{\frac{2\pi i w_i}{h}} x_i.$$

Geometrically, a hamiltonian action of H is a lifting of a rotation action of a base of a fibration $W : \mathbb{C}^K \to \mathbb{C}$. Therefore a time 1 flow restricts to each fiber. More precisely, W satisfies

$$W(t^{w_1}\cdot x_1,\dots t^{w_n}\cdot x_n) = t^h W(x_1,\dots,x_n).$$

In particular, we have

$$W(e^{\frac{2\pi i w_1}{h}s}\cdot x_1,\dots e^{\frac{2\pi i w_n}{h}s}\cdot x_n) = e^{2\pi i s}$$

which means that the flow of H acts as a circle action of an S^1 family of Milnor fiber $W = e^{2\pi i s}$. Set $s = 1$ then we get a desired automorphism. Furthermore, the monodromy flow restricts to a singular fiber $W^{-1}(0)$ and its boundary link

$$L_{W,\delta} := W^{-1}(0) \cap S^{2n-1}_{\delta}.$$

We choose symplectic form ω and a Liouville form λ on $\mathbb{C}^k$ as

$$\omega = \sum_k \frac{1}{2\pi i w_k} dz_k \wedge d\overline{z}_k, \quad \lambda = \sum_k \frac{i}{4\pi w_k}(z_k d\overline{z}_k - \overline{z}_k dz_k). \tag{8.2.1}$$

Then the monodromy flow $\Phi_W(s)$ becomes a Reeb flow $\mathscr{R}$ on $L_{W,\delta}$, where the contact one form is given by a restriction of λ. Starting from any point $x \in L_{W,\delta}$, we get a Reeb chords

$$\gamma : [0,1] \to \partial_\infty W^{-1}(0), \quad \gamma(0) = x, \quad \gamma(1) = \Phi(x)$$

This is an **orbits** of a free quotient $\left(L_{W,\delta}/\langle\Phi\rangle\right)$. Therefore a space of time-1 Reeb orbits of the quotient is a total space $L_{W,\delta}/\langle\Phi\rangle$.

Although we haven't defined full-fledged orbifold symplectic cochains in general, the idea of [CFHW96] [Sei06a] and [KvK16] still works. The ciritical set of action fuctional is a total space $L_{W,\delta}/\langle\Phi\rangle$. A local Floer cohomology $CF^\bullet_{loc}(L_{W,\delta}/\langle\Phi\rangle, H)$ is isomorphic to its Morse cohomology. Notice that the Reeb flow we are using is complex linear on the nose. No non-trivial local system needs to be introduced.

Definition 8.2.2. *A Reeb orbit $\Gamma_W \in CF^\bullet_{loc}(L_{W,\delta}/\langle\Phi\rangle, H)$ is defined to be a cocyle corresponds to a fundamental class*

$$\Gamma_W \leftrightarrow \left[L_{W,\delta}/\langle\Phi\rangle\right] \in H^\bullet\left(L_{W,\delta}/\langle\Phi\rangle; \mathbb{Z}\right).$$

Although the fibration $W : \mathbb{C}^n \to \mathbb{C}$ is singular at the origin, its restriction

$$W|_{S^{2n-1}} : S^{2n-1} \to \mathbb{C}$$

does not. We can canonically identify $L_{W,\delta}$ and $\partial M_{W,cpt} \times \{1\} \subset M_W$. The Reeb orbit Γ_W becomes an hamiltonian orbit, still denoted by Γ_W, of a quotient orbifold $W^{-1}(1)/\langle\Phi\rangle$. Notice that a Φ is always an element of a maximal symmetry group G_W. Therefore, we get an analogous Hamiltonian orbits Γ_W of $[M_W/G_W]$.

It is still possible to check that Γ_W is "closed" in a suitable sense.

Lemma 8.2.3. *Suppose H_{S^1} is G-equivariant, $H_{S^1} > 0$, and C^2-small Morse perturbation of H inside a compact region. Then there is no smooth pseudo-holomorphic cylinder satisfying*

$$u : S^1 \times \mathbb{R} \to [M_W/G_W] \quad \lim_{s\to\infty} u(t,s) = \Gamma_W(t).$$

whose output

$$\gamma_-(t) := \lim_{s\to-\infty} u(s,t)$$

does not lie in $\left(L_{W,\delta}/G_W\right) \times \{1\}$.

Proof. At first, we can rule out the case when the output is outside a compact region using the idea of a spectral sequence [Sei06a] associated to an action filtration. We explain what is going on. For a non-trivial orbit $\gamma \in \mathcal{O}(H_{S^1})$ at the end, its action is given by

$$A_{H_{S^1}}(\gamma) := -\int_{S^1} \gamma^*\lambda + \int_0^1 H_{S^1}(\gamma(t))dt \tag{8.2.2}$$

$$= -2\int_0^1 r^2 dt + \int_0^1 H(\gamma(t))dt + \int_0^1 F(\gamma(t))dt, \quad (H_{S^1} = H(r) + F(r,t)) \tag{8.2.3}$$

$$= -\int_0^1 r^2 + \epsilon \quad (\epsilon << 1) \tag{8.2.4}$$

Nontrivial Hamiltonian orbits are appears as a small perturbation of orbits of level n, which means that it is a perturbation of Hamiltonian orbits $\gamma' \subset \left(L_{W,\delta}/G_W\right) \times \{n\}$ of H. An action value of such orbit is dominated by $-n^2$. The orbit $\Gamma_W(t)$ is an orbit of level 1.

Since $H_{S^1} > 0$, a topological energy of u

$$E_{top}(u) := \int_{S^1\times\mathbb{R}} \omega - d(u^* H_{S^1} \cdot dt) = A_{H_{S^1}}(\gamma_-) - A_{H_{S^1}}(\Gamma_W)$$

must be positive. Therefore, the output γ_- cannot be an orbit of level $n \geq 2$.

Suppose γ_- is an orbit inside a compact region, a Morse critical point of H. since u provides a homotopy class of orbifold loops, we must have $\Phi(\gamma_-(0)) =$

$\gamma_-(1)$. It means that γ_- must be a fixed point of a monodromy Γ. It is impossible because the only fixed point of Γ is the origin, which does not contained in M_W. □

A degeneration of pseudo-holomorphic curves involving Γ_W may have a cylinder breaking, but we conclude that they always comes in cancelling pairs, and hence does not appear in the equations. In particular, an operation $m^{\Gamma}_{n,F,\phi}$ on $\mathscr{W}\mathscr{F}([M_W/G_W])$ can be used to define the desired A_∞-structure.

Definition 8.2.4. *A Fukaya category of a Landau-Ginzburg pair* (W, G_W) *is defined to be*

$$\mathscr{F}(W, G_W) := \mathscr{C}_{\Gamma_W}.$$

For general LG orbifold, we make the following definition (cf. Berglund-Henningson [BH95], Seidel [Sei15]).

Definition 8.2.5. *For any subgroup* $G' < G_W$*, we define* $(G')^T = Hom(G_W/G', \mathbb{C}^*)$. *We define the* $\mathbb{Z}/2$*-graded Fukaya category of the pair* (W, G') *to be the semi-direct product.*

$$\mathscr{F}(W, G') := \mathscr{C}_{\Gamma_W} \rtimes (G')^T$$

For the maximal group $G' = G_W$, the Fukaya category is the same as the one constructed above.

$$\mathscr{F}(W, G_W) = \mathscr{C}_{\Gamma_W}.$$

We will see in the next section that for invertible curve singularities, the Mirror of the monodromy action is given by the restriction of LG model to a hypersurface in Theorem 9.2.1, and expected to hold in general dimensions

Chapter 9

Berglund-Hübsch HMS for curve singularity

In this section, we finally state and prove homological mirror symmetry for Berglund-Hübsch pairs of invertible curve singularities.

Let W be one of the invertible curve singularities. In the previous section, we have defined new A_∞-category $\mathscr{C}_{\Gamma_W}$ using wrapped Fukaya category of the Milnor fiber of W, and quantum cap action of monodromy orbit Γ_W. We first prove

Theorem 9.0.1. *There is a geometric A_∞-functor*

$$\mathscr{G}^{\mathbb{L}} : \mathscr{F}(W, G_W) \to \mathscr{MF}(W^T)$$

which gives a derived equivalence.

This also proves the full version of homological mirror symmetry between Berglund-Hübsch pairs.

Corollary 9.0.2 (Berglund-Hübsch HMS)**.** *For any subgroup $G' < G$, we have the following derive equivalence of $\mathbb{Z}/2$-graded categories*

$$\mathscr{F}(W, G') \cong \mathscr{MF}^{(G')^T}(W^T)$$

where the latter is $(G')^T$-equivariant matrix factorization category of W^T.

The corollary can be deduced from the main theorem, as both sides can be written as semi-direct products, and the functor can be shown to be equivariant ([CHL17]).

A proof of Theorem 9.0.1 occupies the rest of the section. Recall that we have constructed the following HMS for Milnor fibers using localized mirror functor $\mathscr{F}^{\mathbb{L}}$ in Theorem 7.0.1.

$$\mathscr{W}\mathscr{F}([M_W/G_W]) \xrightarrow{\mathscr{F}^{\mathbb{L}}} \mathrm{MF}(W^T + xyg) \ ,$$

where a polynomial g was given by

$$\mathrm{Fermat}: g = z, \ \ \mathrm{Chain}: g = (z - y^{q-1}), \mathrm{Loop}: g = (z - x^{p-1} - y^{q-1}).$$

The proof consists of the following two theorems. At first, we compute the class Γ_W explicitly and show that it is a mirror to $\cdot g$.

Theorem 9.0.3. *There is a diagram of A_∞-bimodules*

$$\begin{tikzcd}
\mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r, "\cap\Gamma_W"] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{F}(W, G_W) \arrow[r] \arrow[d, "\widetilde{\mathscr{F}}^{\mathbb{L}}"] & {} \\
\mathrm{MF}(W^T + xyg) \arrow[r, "\cdot g"] & \mathrm{MF}(W^T + xyg) \arrow[r] & \mathrm{MF}(W^T) \arrow[r] & {}
\end{tikzcd}$$

whose all vertical lines are quasi-isomorphisms.

Next, we enhance this equivalence to that of A_∞ categories.

Theorem 9.0.4. *There exist an A_∞-functor, $\mathscr{G}^{\mathbb{L}}$, extending the bimodule map $\widetilde{\mathscr{F}}^{\mathbb{L}}$*

$$\mathscr{G}^{\mathbb{L}} : \mathscr{F}(W, G_W) \to \mathscr{M}\mathscr{F}(W^T)$$

which gives a derived equivalence.

9.1 Computation of Γ_W

We compute Γ_W for invertible curve singularities. The boundary $\partial M_{W,cpt}$ is a union of circle and G_W acts on them by rotation. In this particular case, Γ_W is represented by a union of loops around punctures.

Proposition 9.1.1. *A class Γ_W is given by a sum of hamiltonian orbits, geometrically represented by the following element of $\pi_1\left(\mathbb{P}^1_{a,b,c}\right)$ respectively.*

1. ***Fermat type*** $\left(\simeq \mathbb{P}^1_{p,q,\frac{pq}{gcd(p,q)}}\right)$*:* $\Gamma_W \leftrightarrow (\gamma_3)^{-1}$*;*

2. ***Chain type*** $\left(\simeq \mathbb{P}^1_{pq,q,\frac{pq}{gcd(p-1,q)}}\right)$*:* $\Gamma_W \leftrightarrow (\gamma_1)^{1-p} + (\gamma_3)^{-1}$*;*

3. ***Loop type*** $\left(\simeq \mathbb{P}^1_{pq-1,pq-1,\frac{pq}{gcd(p-1,q-1)}}\right)$*:* $\Gamma_W \leftrightarrow (\gamma_1)^{1-p} + (\gamma_2)^{1-q} + (\gamma_3)^{-1}$.

Proof. The idea is that locally around zero, $W^{-1}(c)$ shares the same coordinate system near the punctures. We will describe an orbit of an induce circle action around the puncture. Those orbits are tranversally nondegenerate, appears as an S^1 family of orbit. Γ_W corresponds to a fundamental class of such S^1-family.

1. **Fermat type** $x^p + y^q$
   Recall that the weight of this polynomial is $(pq; q, p)$ so the Reeb flow is

$$(x, y) \mapsto (e^{\frac{2\pi i t}{p}} x, e^{\frac{2\pi i t}{q}} y).$$

   The quotient M_W/G has a single puncture at infinity. Under a coordinate change

$$X = \frac{1}{x}, \quad Y = \frac{1}{y}$$

   the equation for $W^{-1}(c)$ becomes $X^p + Y^q = cX^pY^q$. It is a Riemann surface of a mutivalued function

$$Y = \left(\frac{X^p}{cX^p - 1}\right)^{\frac{1}{q}}.$$

The only brach point we should consider is that of 0. A local coordinate chart near the puncture is given by

$$w^{\frac{q}{gcd(p,q)}} = X = \frac{1}{x}$$

and the induced flow is given by

$$w \mapsto e^{-\frac{2\cdot gcd(p,q)\pi i t}{pq}} \cdot w$$

Its winding number around ∞ is -1.

2. **Chain type** $x^p + xy^q$
The weight of this polynomial is $(q, p-1, pq)$ and Reeb flow is

$$(x, y) \mapsto (e^{\frac{2\pi i t}{p}} x, e^{\frac{2(p-1)\pi i t}{pq}} y).$$

The quotient M_W/G_W has two punctures at 0 and and ∞. Around $x = 0$, $W^{-1}(c)$ is a Riemann surface of a function

$$\frac{1}{y} = Y = \left(\frac{x}{c - x^p}\right)^{\frac{1}{q}}$$

with branch points 0 and $(\xi_p)^k c^{\frac{1}{p}}$ where ξ_p is a p-th root of unity. Notice that brach points are converging to 0 as $c \to 0$. We want to find a local chart for $c \to 0$, so we choose lines connecting 0 and $(\xi_p)^k c^{\frac{1}{p}}$ $(k \neq 0)$ as branch cuts. Consider a small loop inside an x-plane encircling 0 and c both. It sends $(x, y) \mapsto (x, e^{\frac{2q\pi i}{p-1}} y)$. Therefore y^{-1} is a function of $x^{\frac{1-p}{q}}$ locally around $x = 0$. A local coordinate chart near 0 and induced flow will be

$$(w_1)^q = x^{1-p}, \quad w_1 \mapsto e^{\frac{-2(p-1)\pi i t}{pq}}$$

The winding number of a corresponding time-1 orbit is $(1-p)$. At ∞, we can do the same thing as we did for Fermat type. Under a coordinate change

$$X = \frac{1}{x}, \quad Y = \frac{1}{y}$$

the equation for $W^{-1}(c)$ becomes $X^{p-1} + Y^q = cX^pY^q$. It is a Riemann surface of a mutivalued function

$$Y = \left(\frac{X^{p-1}}{cX^p - 1}\right)^{\frac{1}{q}}.$$

Therefore, a local coordinate chart near the puncture and induced flow becomes

$$(w_2)^{\frac{q}{gcd(p-1,q)}} = X = \frac{1}{x}, \quad w_2 \mapsto e^{\frac{-2\cdot gcd(p-1,q)\pi it}{pq}} \cdot w_2$$

A winding number around ∞ is -1.

3. **Loop type** $x^py + xy^q$
The weight of this polynomial is $(q-1, p-1, pq-1)$. Reeb flow is given by

$$(x, y) \mapsto (e^{\frac{2(q-1)\pi it}{pq-1}}x, e^{\frac{2(p-1)\pi it}{pq-1}}y).$$

The quotient M_W/G_W has three punctures at 0 ,1 and ∞. The following reparametrization presents $W^{-1}(c)$ as Riemann surface of functions of z.

$$x = \left(\frac{z^q}{c-z}\right)^{\frac{1}{pq-1}}, \quad y = \left(\frac{(c-z)^p}{z}\right)^{\frac{1}{pq-1}}$$

To find a local coordinate near $x = 0$ when $c \to 0$, choose a line connecting 0 and c inside a z-plane as a branch cut. Again, consider a small loop inside a z-plane around 0 and c both. It sends $(x, y) \mapsto (e^{\frac{2(q-1)\pi i}{pq-1}}x, e^{\frac{2(p-1)\pi i}{pq-1}}y)$. Therefore y^{-1} is a function of $x^{\frac{1-p}{q-1}}$ locally around $x = 0$. Therefore, a local coordinate and induced flow will be

$$(w_1)^{q-1} = x^{1-p}, \quad w_1 \mapsto e^{\frac{2(1-p)\pi it}{pq-1}}$$

The winding number of a corresponding time-1 orbit is $(1-p)$. The rest of the procedure is entirely the same.

□

9.2 Mirror of the Monodromy action: Restriction of LG model to a hypersurface

On the symplectic side, we will consider the monodromy Γ_W-action(quantum cap action in 3.3.3) to define the new A_∞-category $\mathscr{C}_{\Gamma_W}$, and on the complex side, we will consider the restriction to the hypersurface $g(x, y, z) = 0$.

Proposition 9.2.1. *The following diagram*

$$\begin{tikzcd} \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r, "\cap\Gamma_W"] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[d, "\mathscr{F}^{\mathbb{L}}"] \\ \mathrm{MF}(\widetilde{W}) \arrow[r, "g"] & \mathrm{MF}(\widetilde{W}) \end{tikzcd}$$

commutes up to homotopy H. More precisely, we have pre-homomorphism of A_∞-bimodules $H^{\mathbb{L}} = \{H^{\mathbb{L}}_k\}_{k=1}^\infty$, satisfying

$$(H^{\mathbb{L}} \circ m \pm D \circ H^{\mathbb{L}})(a_1, \ldots, \underline{b}, \ldots, a_n) \tag{9.2.1}$$

$$= \sum_{j \le k, j+l \ge k} \mathscr{F}^{\mathbb{L}}_{n-l+1}(a_1, \ldots, \cap\Gamma_W(a_{j+1}, \ldots, \underline{b}, \ldots, a_{j+l}), \ldots, a_n) \tag{9.2.2}$$

$$\pm g \cdot (\mathscr{F}^{\mathbb{L}}_{n+1}(a_1, \ldots, \underline{b}, \ldots, a_n)) \tag{9.2.3}$$

Proof. Recall $\boldsymbol{b} = xX = yY + zZ$ denotes a bounding cochain of $\mathbb{L}$. Here, we use bold font for $\boldsymbol{b}$ in this section to emphasize its role, and also denote the component of cap action by $(\cap\Gamma_W)_l = N_l$ for simplicity.

Definition 9.2.2. *Define a pre-homomorphism of bimodules $H^{\mathbb{L}} = \{H^{\mathbb{L}}_k\}_{k=1}^\infty$ as*

$$H^{\mathbb{L}}_{n+1}(a_1, \ldots, a_k, \underline{b}, a_{k+1}, \ldots, a_n)(x) = \sum_l N_{n+l+2}(\mathbf{b}, \ldots, \mathbf{b}, x, a_1, \ldots, \underline{b}, \ldots, a_n)$$

Also define

$$\mathscr{H}^{\mathbb{L}}_0(x) = \sum_l N_{l+1}(\mathbf{b}, \ldots, \mathbf{b}, \underline{x}), \quad x \in CW(\mathbb{L}, L_1)$$

Notice that $\mathscr{H}^{\mathbb{L}}_0$ is not a part of a bimodule map. Consider a boundary of 1-dimensional components of moduli space governing $\mathscr{H}^{\mathbb{L}}(a_1, \ldots, \underline{b}, \ldots, a_n)$. There are four different possible degeneration

- (string of $\boldsymbol{b}$s is not broken, the interior point remains) This components contribute to

$$\sum_{j\leq k, j+l\geq k} \mathscr{F}^{\mathbb{L}}_{n-l+1}(a_1,\ldots,N_{l+1}(a_{j+1},\ldots,\underline{b},\ldots,a_{j+l}),\ldots,a_n).$$

- (string of $\boldsymbol{b}$s is not broken, the interior point escapes towards special output) This components contribute to

$$H^{\mathbb{L}}\circ m(a_1,\ldots,\underline{b},\ldots,a_n)=\sum H^{\mathbb{L}}_{n-l+1}(a_1,\ldots,m_{l+1}(a_{j+1},\ldots,\ldots,a_{j+l}),\ldots,a_n)$$

  Notice that since the interior point escaped toward the special output, the special input $\underline{b}$ is no more special w.r.t m_k operation. it can be anywhere.

- (string of $\boldsymbol{b}$s is broken, the interior point remains) This components contributes to

$$D\circ H^{\mathbb{L}}=[m_1^{\mathbf{b}},H^{\mathbb{L}}] \text{ and } \widetilde{W}\cdot H^{\mathbb{L}}(a_1,\ldots,\underline{b},\ldots,a_n)$$

- (string of $\boldsymbol{b}$s is broken, the interior point escapes toward special output) This components contributes to

$$\mathscr{H}^{\mathbb{L}}_0\cdot(\mathscr{F}^{\mathbb{L}}_{n+1}(a_1,\ldots,\underline{b},\ldots,a_n))$$

Notice that in the category of matrix factorization, the element $(\widetilde{W})$ acts as a zero. Therefore, to prove 9.2.1, we have to show

$$\mathscr{H}^{\mathbb{L}}_0(\alpha)=g\cdot\alpha,\quad,\forall\alpha\in Hom(\mathbb{L},L),\quad\forall L\in Ob(\mathscr{W}\mathscr{F}([M_W/G_W]))$$

For this, we need to recall Kodaira-Spencer map, which is special case of the closed-open map. Closed-open map refers to a map from closed string theory to open string theory, and more concretely, for a closed (resp. open) symplectic manifold M, the following maps are expected to be ring isomorphisms.

$$QH^\bullet(M)\to HH^\bullet(\mathscr{F}(M))\big(\text{resp. } SH^\bullet(M)\to HH^\bullet(\mathscr{W}\mathscr{F}(M))\big)$$

We first recall the known results related to our construction. Fukaya-Oh-Ohta-Ono [FOOO16] constructed a Kodaira-Spencer map $QH^*(M) \to Jac(W)$ by counting holomorphic discs with interior insertion of a quantum cohomology class with Lagrangian boundary condition. Such construction was generalized to $\mathbb{P}^1_{a,b,c}$ by the first author with Amorim, Hong and Lau [ACHL20], which we will adapt to our cases at hand.

When the output image of a closed-open map is a multiple $c[L]$ of fundamental class of Lagrangian, this coefficient c suitably decorated with deformation variables provide such a map. For a Liouville domain M, closed-open map from symplectic cohomology was first introduced by Seidel [Sei06b], and it plays a crucial role in Abouzaid's work on generation criterion of wrapped Fukaya category [Abo10]. The first of these map is given by

$$CO_0 : SH^\bullet(M) \to HW^\bullet(L, L)$$

Pascaleff [Pas19] proved that for the complement of normal crossing anti-canonical divisor, and for a Lagrangian section L, CO_0 gives isomorphism in degree zero. On the other hand, Tonkonog [Ton19] found quite interesting relationship between potential functions of Lagrangians on Fano manifolds and the symplectic cohomology ring of the smooth anti-canonical complement.

In our case, we apply these ideas to the Seidel Lagrangian $\mathbb{L}$ in the quotient $[M_W/G_W]$.

Definition 9.2.3. *Kodaira-Spencer map*

$$KS^{\mathbf{b}} : SH^{even}([M_W/G_W]) \to \mathrm{Jac}(\widetilde{W})$$

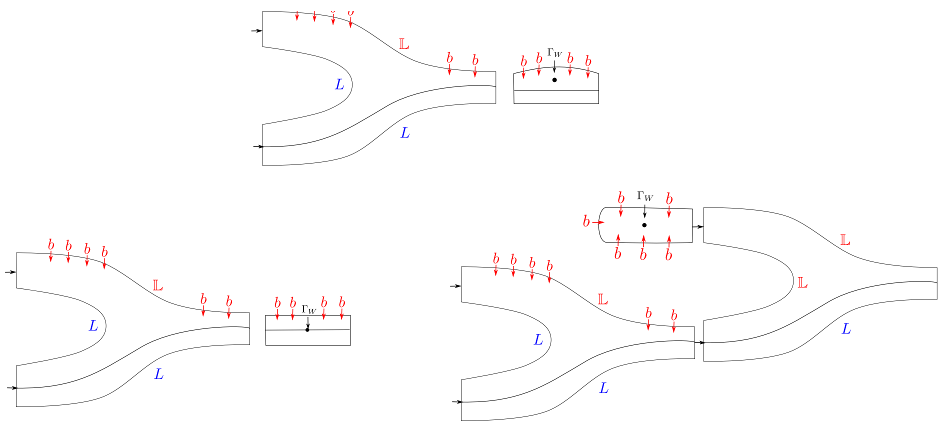
is defined by the reading the coefficient of $[\mathbb{L}]$ *of the output of closed-open map given by an interior insertion of a symplectic cohomology class and boundary insertion of* $\mathbf{b}$*'s.*

The fact that this map is well-defined can be proved as in [ACHL20], and we omit the details.

Proposition 9.2.4. *Up to homotopy,*

$$\mathscr{H}_0^{\mathbb{L}}(\alpha) = KS^{\mathbf{b}}(\Gamma_W) \circ \alpha.$$

Proof. Consider a moduli space of pseudo-holomorphic curves governing $\mathscr{H}_0^{\mathbb{L}}$, but we allow its interior point to move towards boundary on $\mathbb{L}$. An operation associated to the moduli space provides a homotopy between $\mathscr{H}_0^{\mathbb{L}}(\alpha)$ and $\mathrm{KS}^{\mathbf{b}}(\Gamma) \circ \alpha$. See Figure 9.1. □



$\mathcal{H} \circ \mathcal{F}^{\mathbb{L}}(-)$ $\quad$ $KS^b(\Gamma_W) \times \mathcal{F}^{\mathbb{L}}(-)$

Figure 9.1: Kodaira-Spencer invariant appears.

Proposition 9.2.5.

$$KS^{\mathbf{b}}(\Gamma_W) = g \cdot 1$$

Proof. We should count pseudo-holomorphic discs whose boundary lies in Seidel's Lagrangian $\mathbb{L}$, corners are at immersed intersections of $\mathbb{L}$ and interior puncture asymptotic to a Hamiltonian orbit Γ_W. The $\Omega-$ and H^1-grading of $SH^\bullet$ and $CF^\bullet(\mathbb{L},\mathbb{L})$ must be compatible. It implies that the only possible contribution of $\mathrm{KS}^{\mathbf{b}}(\gamma_i^k)$ is a k-th power of a variable associated to the immersed corner of $\mathbb{L}$ opposite to the puncture. We can see such a polygon in a picture explicitly as in Figure 9.2. A careful sign computation combined with 9.1.1 will calculates $\mathrm{KS}^{\mathbf{b}}(\Gamma)$ explicitly.

1. Fermat type $F_{p,q}$: $\mathrm{KS}^{\mathbf{b}}(\Gamma) = z \cdot 1$
2. Chain type $C_{p,q}$: $\mathrm{KS}^{\mathbf{b}}(\Gamma) = (z - y^{p-1}) \cdot 1$

3. Loop type $L_{p,q}$: $\mathrm{KS}^{\mathbf{b}}(\Gamma) = (z - x^{q-1} - y^{p-1}) \cdot 1$

These polynomials are exactly $g(x,y,z)$ we want.

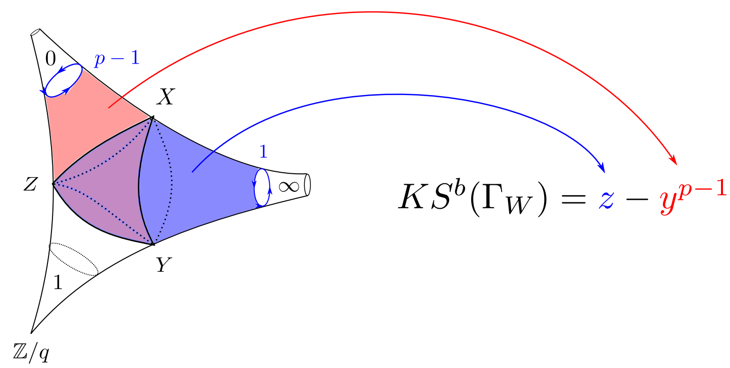


Figure 9.2: $\mathrm{KS}^{\mathbf{b}}(\Gamma_W)$ for a chain type singularity

To prove that there are no other contribution, we use the idea of Tonkonog [Ton19] of domain stretching. Namely, if we consider a compactification of $[M_W/G_W]$ into $\mathbb{P}^1_{a,b,c}$, holomorphic discs that contribute to the closed-open map from $QH^\bullet(\mathbb{P}^1_{a,b,c}) \to Jac(\widetilde{W})$ has been worked out in [ACHL20]. If we interpret Reeb orbits as suitable orbifold insertions, we obtain the above computations. We can relate it to the computation of $CO_0(\Gamma) \in \mathrm{Jac}(\widetilde{W})$ using the construction of Tonkonog. For curve singularities, the hypersurface Σ in [Ton19] is just a point (or points) playing the role of Donaldson hypersurface.

Given a holomorphic disc with boundary on a compact Lagrangian K in the compact space X, Tonkonog introduced a new stretching procedure for holomorphic curves based on domain stretching as in standard Floer theory. By clever choice of sequence of Hamiltonians (called S-shaped) for the domain stretching, the standard J-holomorphic discs breaks into parts which share Reeb orbits as same asymptotics. Reeb orbits for S-shaped Hamiltonian are divided into types I, II, III, IV_a, IV_b, depending on their position with respect to Liouville collar. Key part of the proof is to show that only type II Reeb orbit appears in the breaking. In our case, if breaking occurs at type I, IV_a, IV_b Reeb orbits (which are constant orbits), then collecting the parts from this constant orbit to Σ we get a non-trivial sphere that maps to X. The starting polygon in X do not intersect other vertices of $\mathbb{P}^1_{a,b,c}$, and this intersection number with perturbed J-holomorphic curve is positive, so the sphere should not intersect other vertices. Therefore, such sphere cannot exist. This excludes these type of Reeb orbits as breaking orbits. The argument again type III orbit using no escape lemma still applies to our case. One can see that any disc bubble would increase the intersection with vertices of orbisphere, hence do not occur.

□

The proof of 9.2.1 is now complete. □

We obtain the Theorem 9.0.3 as a corollary.

Corollary 9.2.6. *The following diagram commutes up to homotopy;*

$$\begin{tikzcd}
\mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r, "\cap\Gamma_W"] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{W}\mathscr{F}([M_W/G_W]) \arrow[r] \arrow[d, "\mathscr{F}^{\mathbb{L}}"] & \mathscr{F}(W, G_W) \arrow[r] \arrow[d, "\widetilde{\mathscr{F}^{\mathbb{L}}}"] & {} \\
\mathrm{MF}(W^T + xyg) \arrow[r, "\cdot g"] & \mathrm{MF}(W^T + xyg) \arrow[r] & \mathrm{MF}(W^T) \arrow[r] & {}
\end{tikzcd}$$

where $\widetilde{\mathscr{F}^{\mathbb{L}}} = \begin{pmatrix} \mathscr{F}^{\mathbb{L}} & \mathscr{H}^{\mathbb{L}} \\ 0 & \mathscr{F}^{\mathbb{L}} \end{pmatrix}$. *Each row is a distinguished triangle of bimodules. All vertical lines induces quasi-isomorphisms.*

This establish an equivalence of $\mathscr{F}(W, G_W)$ and $\mathrm{MF}(W^T)$ at the level of bimodules. To state a full mirror symmetry statement, we are going to promote it to an equivalence of A_∞ category.

9.3 Berglund-Hübsch mirror symmetry

We introduce a relevant operation which generalizes $\mathscr{H}_0^{\mathbb{L}}$ in the previous section.

Definition 9.3.1. *Define $\mathscr{H}^{\mathbb{L}}$ as follows;*

$$\mathscr{H}^{\mathbb{L}}_k : \mathrm{Hom}_{\mathscr{F}(W,G_W)}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{F}(W,G_W)}(L_{k-1}, L_k) \to \mathrm{Hom}(M_{L_0}, M_{L_k}).$$

$$\mathscr{H}^{\mathbb{L}}_k(a_1,\ldots,\epsilon b_{i_1},\ldots,\epsilon b_{i_j},\ldots,a_k)(x) = \sum_{l,p} \pm m^{\Gamma_W}_{l+k+2,\{(l+1)+\widehat{F}^p\}\cup(l+1)}(\boldsymbol{b},\ldots,\boldsymbol{b},x,a_1,\ldots,a_k)$$

Here, $F = \{i_1,\ldots,i_j\}$ and $(l+1)+\widehat{F}^p$ means a translation of $\widehat{F}^p$ by $(l+1)$.

Definition 9.3.2. *Let*

$$\mathscr{G}^{\mathbb{L}} : \mathscr{F}(W, G_W) \to \mathrm{MF}(W^T)$$

as a pre-A_∞ functor

$$\mathscr{G}^{\mathbb{L}} = M^{\mathbf{b}} + \mathscr{H}^{\mathbb{L}}_k$$

More precisely, it is defined as

$$\mathscr{G}^{\mathbb{L}}(L) = \mathrm{Cone}\left(\mathscr{F}^{\mathbb{L}}(L) \xrightarrow{g} \mathscr{F}^{\mathbb{L}}(L) \right)$$

and

$$\begin{aligned}
&\left[\mathscr{G}^{\mathbb{L}}_k(a_1,\ldots,\epsilon b_{i_1},\ldots,\epsilon b_{i_j},\ldots,a_k)\right](x)\\
&= \sum_{l} m^{\Gamma_W}_{k+l+1,(l+1)+F}(\boldsymbol{b},\ldots,\boldsymbol{b},x,a_1,\ldots,b_{i_1},\ldots,b_{i_j},\ldots,a_k)\\
&+ \sum_{l,p} m^{\Gamma_W}_{l+k+2,\{(l+1)+\widehat{F}^p\}\cup(l+1)}(\boldsymbol{b},\ldots,\boldsymbol{b},x,a_1,\ldots,b_{i_1},\ldots,b_{i_j},\ldots,a_k)\\
&+ \epsilon \cdot \sum_{l,p} m^{\Gamma_W}_{k+l+1,(l+1)+\widehat{F}^p}(\boldsymbol{b},\ldots,\boldsymbol{b},x,a_1,\ldots,b_{i_1},\ldots,b_{i_j},\ldots,a_k)
\end{aligned}$$

Proposition 9.3.3. *$\mathscr{G}^{\mathbb{L}}$ is an A_∞ functor.*

Proof. The caculation is similar. Consider an 1-dimensinal components of moduli space governing

$$\sum_{l} m^{\Gamma_W}_{k+l+1,(l+1)+F}(\boldsymbol{b},\ldots,\boldsymbol{b},x,a_1,\ldots,b_{i_1},\ldots,b_{i_j},\ldots,a_k).$$

There are two different possible degenerations.

- (string of $\boldsymbol{b}$s is not broken) This components contribute to

$$\sum_{l,1\leq q\leq r\leq k,F_i} m^{\Gamma_W}_{k+l-(r-q),(l+1)+F_1}(\boldsymbol{b},\dots,\boldsymbol{b},x,a_1,\dots,b_{i_1},\dots$$

$$\dots,m^{\Gamma_W}_{r-q+1,(l+1)+F_2}(a_q,\dots,a_r)\dots,b_{i_j},\dots,a_k).$$

  Here, F_i are possible admissible cuts of F. It corresponds to a

$$\mathscr{G}^{\mathbb{L}}(a_1,\dots,M(\dots),\dots,a_k).$$

- (string of $\mathbf{b}$s is broken) This components contribute to

$$\sum_{l,l_1+l_2=l,q\leq k,F_i} m^{\Gamma_W}_{k+l_1+1-r,(l+1)+F_1}(\boldsymbol{b},\dots,\boldsymbol{b},m^{\Gamma_W}_{l_2+r+1,(l_1+1)+F_2}(\boldsymbol{b}\dots,\boldsymbol{b},x,a_1,\dots$$

$$\dots,b_{i_1},\dots,a_q,\dots,a_r)\dots,b_{i_j},\dots,a_k).$$

  Here, F_i are possible admissible cuts of F. It corresponds to a

$$\mathscr{G}^{\mathbb{L}}(\dots)\circ\widetilde{\mathscr{G}}(\dots) \text{ or } [m_1^{\boldsymbol{b}},\mathscr{G}^{\mathbb{L}}(\dots)].$$

$\square$

Finally, we get

Corollary 9.3.4. *The functor*

$$\mathscr{G}^{\mathbb{L}}:\mathscr{F}(W,G_W)\to\mathscr{MF}(W^T)$$

is an A_∞ equivalence which fits into a diagram

$$\begin{array}{ccccccc}
\mathscr{WF}([M_W/G_W]) & \xrightarrow{\cap\Gamma_W} & \mathscr{WF}([M_W/G_W]) & \longrightarrow & \mathscr{F}(W,G_W) & \longrightarrow \\
\downarrow{\scriptstyle \mathscr{F}^{\mathbb{L}}} & & \downarrow{\scriptstyle \mathscr{F}^{\mathbb{L}}} & & \downarrow{\scriptstyle \mathscr{G}^{\mathbb{L}}} & \\
\mathrm{MF}(W^T+xyg) & \xrightarrow{g} & \mathrm{MF}(W^T+xyg) & \longrightarrow & \mathrm{MF}(W^T) & \longrightarrow
\end{array}$$

Proof. Equivalence statement follows directly from a bimodule level computation we did in the last section. Next, simply notice that the first-order component of $\mathscr{G}^{\mathbb{L}}$ coincides with $\widetilde{\mathscr{F}}^{\mathbb{L}}$ is the last subsection. $\square$

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국문초록

이 논문에서는 리우빌 다양체 M의 사교 코호몰로지 군의 원소 $\Gamma \in SH^\bullet(M)$ 가 주어져 있을 때, Γ 의 양자 곱 작용 (quatum cap action) $\Gamma : CW^\bullet(L,L) \to CW^\bullet(L,L)$ 이 호모토피적으로 사라지는 새로운 호모토피 결합 범주 (A_∞-category) $\mathscr{C}_\Gamma$ 를 건설하고자 한다.

이 새로운 건설법을 바탕으로 하여 가중 동차 다항식 W과 그것의 대칭군 G로 이루어진 사교 란다우-긴즈버그(Landau-Ginzburg) 모델 (W,G) 을 만든다. 밀너 올(MIlnor fiber) 의 감긴 푸카야 범주 (wrapped Fukya category) 와 그것에 작용하는 모노드로미 작용 (monodromy action) 을 사용하여, 모노드로미 작용이 사라지는 새로운 범주 $\mathscr{F}(W,G)$를 만든다. 이것은 고전적인 특이점 이론의 변분 연산자(variation operator)의 사교기하적 유추로 간주할 수 있다.

이에 더해, 모노드로미 작용의 거울 현상이 거울 란다우-긴즈버그 모델을 특정한 초곡면에 제한시키는 것임을 보인다. 그것의 응용으로, 모든 가역 곡선 특이점에 대해 버글룬드-휼스 추측을 증명한다.

주요어휘**:** 라그랑지언 플로어 이론, 거울대칭, 오비폴드, 가역 다항식, 행렬 인수분해

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